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Partial differential equations

Blow-up of solutions to a semilinear heat equation with a viscoelastic term and a nonlinear boundary flux ☆

Explosion de solutions de l'équation de la chaleur semi-linéaire avec un terme viscoélastique et un flux de limite non linéaire

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A R T I C L E I N F O

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A B S T R A C T

In this article, we study a semilinear heat equation

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s = 0$$

with a viscoelastic term and a nonlinear flux on the boundary. By defining a modified energy functional and using a concavity argument, a blow-up result for solutions with negative initial energy is proved.

R É S U M É

Dans cet article, on étudie une équation de la chaleur semi-linéaire

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s = 0$$

avec un terme viscoélastique et un flux non linéaire sur la limite. En définissant une modifiée fonctionnelle d'énergie et en utilisant un argument de concavité, un résultat d'explosion des solutions avec énergie initiale négative est prouvé.

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1. Introduction

In this paper, we investigate the blow-up properties of solutions to the following semilinear initial boundary value problem

$$\begin{cases} u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u(x,s)}{\partial \nu}\,\mathrm{d}s = |u|^{p-2}u, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where g: $\mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function, $p > 2$, $\Omega \subset \mathbb{R}^N (N \ge 2)$ is a bounded domain with smooth boundary $\partial\Omega$ and ν is the unite outward normal on $\partial\Omega$.

Problem (1.1) arises from a variety of mathematical models of applied sciences such as viscoelastic fluids and electro-rheological fluids ([1,15]). In particular, when we study heat conduction in materials with memory, the classical Fourier law of heat flux is usually replaced by the following form

$$q = -d\nabla u - \int_{-\infty}^{t} \nabla[k(x,t)u(x,s)]\,\mathrm{d}s, \tag{1.2}$$

where u is the temperature, $d > 0$ is the diffusion coefficient and the integral term represents the memory effect of the material. In the past few decades, the study of equations with memory terms has drawn a considerable attention and many results have been obtained on the existence, uniqueness and regularities of weak or classical solutions. We refer the interested readers to [18,19]. From a mathematical point of view, it is hoped that the integral term is dominated by the leading term in the equation and therefore the theory of parabolic equation can be applied to Problem (1.1).

Finite-time blow-up is one of the most important properties of many evolutionary equations. When one needs to determine whether the solutions of a given problem blow up in finite time or not, there are many methods to use. For example, the first eigenvalue method introduced by Kaplan in 1963, the concavity method introduced by Levine in the 1970s, and the comparison method based on the comparison principle. In the absence of the memory term ($g(t) \equiv 0$), properties including blow-up time, blow-up rate, blow-up set and blow-up profiles of solutions to semilinear equations like (1.1) have been investigated by many authors, see [7–10,17] for equations with sources in the interior and [5,6,11,16] for those with sources on the boundary.

However, when $g \not\equiv 0$, the blow-up results are much fewer. Messaoudi [12] studied

$$\begin{cases} u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s = |u|^{p-2}u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

and showed that the solutions to (1.3) blow up in finite time when, among other conditions, the initial energy $E(0) \le 0$, where

$$E(t) = \frac{1}{2}(g \diamond \nabla u)(t) + \frac{1}{2}\Big(1 - \int_0^t g(s)\mathrm{d}s\Big)\|\nabla u(t)\|_2^2 - \frac{1}{p}\|u(t)\|_p^p, \tag{1.4}$$

and

$$(g \diamond v)(t) = \int_0^t g(t-s)\|v(x,t) - v(x,s)\|_2^2\,\mathrm{d}s. \tag{1.5}$$

The idea of the proof is to establish a connection between $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2$ and $\|u\|_2^r$ for some $r > 1$. Later, Messaoudi [13] and Fang et al. [4] also proved that the solutions to (1.3) may blow up in finite time when the initial energy is positive, but suitably small.

Motivated by the works mentioned above, we shall study the blow-up properties of solutions to (1.1). However, the situations are more complicated. On one hand, the connection between $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2$ and $\|u\|_2^r$, as was established in [12], cannot be obtained here since the source is on the boundary; on the other hand, the appearance of the relaxation function g makes the powerful comparison principle invalid. To overcome the above two difficulties, we shall define a modified energy functional and use a concavity argument to establish a blow-up criterion. The main result of this paper and its proof will be given in Section 2.

2. Main result

In order to state and prove the main result, we first define the modified energy functional

$$E(t)=\frac{1}{2}(g\diamond\nabla u)(t)+\frac{1}{2}\Big(1-\int_0^t g(s)\mathrm{d}s\Big)\|\nabla u(t)\|_{2,\Omega}^2-\frac{1}{p}\|u(t)\|_{p,\partial\Omega}^p, \tag{2.1}$$

where $(g\diamond u)(t)$ is given in (1.5). For simplicity, we shall omit Ω and $\partial\Omega$ in the subscript of the norms in the rest of this paper. Throughout this paper, the relaxation function g and the parameter p are supposed to satisfy

$$g(s)\geq 0,\qquad g'(s)\leq 0,\qquad 1-\int_0^\infty g(s)\,\mathrm{d}s=l>0, \tag{2.2}$$

and

$$2<p\leq\frac{2N-3}{N-2},\qquad N>3;\qquad p>2,\qquad N=2. \tag{2.3}$$

Before stating our main result, we first give a definition for a strong solution to (1.1), see [14]. The local existence of such a strong solution can be obtained by applying the Galerkin method as in [3] and the details are omitted.

Definition 2.1. A strong solution to Problem (1.1) is a function $u\in C([0,T);H^1(\Omega))\cap C^1([0,T);L^2(\Omega))$, satisfying

$$\int_0^t\int_\Omega\Big(u_t\phi+\nabla u\nabla\phi-\int_0^s g(s-\tau)\nabla u(\tau)\nabla\phi(s)\mathrm{d}\tau\Big)\mathrm{d}x\,\mathrm{d}s=\int_0^t\int_{\partial\Omega}|u|^{p-2}u\phi\,\mathrm{d}\sigma\,\mathrm{d}s$$

for all $t\in[0,T)$ and all $\phi\in C([0,T);H^1(\Omega))$.

By the Sobolev trace imbedding theorem (Theorem 5.36 in [2]), we know that condition (2.3) implies $|u|^{p-2}u\in L^2(\partial\Omega)$ and hence $\int_{\partial\Omega}|u|^{p-2}u\phi\,\mathrm{d}\sigma$ makes sense. Condition (2.2) is necessary to guarantee the parabolicity of (1.1).

By multiplying the equation in (1.1) by u_t, integrating by parts over Ω, one obtains that

$$\begin{aligned}\int_\Omega u_t^2\,\mathrm{d}x&=\int_\Omega\Delta uu_t\,\mathrm{d}x-\int_\Omega u_t\Big(\int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s\Big)\,\mathrm{d}x\\&=\int_{\partial\Omega}u_t\Big(\frac{\partial u}{\partial\nu}-\int_0^t g(t-s)\frac{\partial u}{\partial\nu}\mathrm{d}s\Big)\mathrm{d}\sigma-\int_\Omega\nabla u\nabla u_t\,\mathrm{d}x\\&\quad+\int_\Omega\Big(\int_0^t g(t-s)\nabla u(s)\nabla u_t(t)\mathrm{d}s\Big)\mathrm{d}x\\&=-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_\Omega|\nabla u|^2\mathrm{d}x\Big)+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{p}\int_{\partial\Omega}|u|^p\mathrm{d}\sigma\Big)+\int_0^t g(t-s)\int_\Omega\nabla u(s)\nabla u_t(t)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

The last term of the right-hand side of the above equality can be rewritten as

$$\begin{aligned}&\int_0^t g(t-s)\int_\Omega\nabla u(s)\nabla u_t(t)\,\mathrm{d}x\,\mathrm{d}s\\&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t g(t-s)\int_\Omega|\nabla u(s)-\nabla u(t)|^2\,\mathrm{d}x\,\mathrm{d}s\Big)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t g(s)\,\mathrm{d}s\int_\Omega|\nabla u(t)|^2\mathrm{d}x\Big)\\&\quad+\frac{1}{2}\int_0^t g'(t-s)\int_\Omega|\nabla u(s)-\nabla u(t)|^2\,\mathrm{d}x\,\mathrm{d}s-\frac{1}{2}g(t)\int_\Omega|\nabla u(t)|^2\,\mathrm{d}x.\end{aligned}$$

Combining the above two equalities with (2.1), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\int_\Omega u_t^2\,\mathrm{d}x - \frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x + \frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)|^2\,\mathrm{d}x\,\mathrm{d}s \le 0 \tag{2.4}$$

for smooth solutions. The same result can be established for strong solutions and for almost every t, by a standard density argument. The main result of this paper is the following.

Theorem 2.1. *Suppose that (2.2)–(2.3) hold and that $u_0 \in H^1(\Omega)$ satisfying $E(0) < 0$. If*

$$\int_0^\infty g(s)\,\mathrm{d}s < \frac{p-2}{p-3/2}, \tag{2.5}$$

then any strong solution of (1.1) blows up in finite time.

Proof. The theorem will be proved by using a concavity argument introduced by Levine. For this, set

$$I(t) = \int_0^t \int_\Omega u^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + A, \tag{2.6}$$

where $A > 0$ is a constant to be fixed. Then

$$I'(t) = \int_\Omega u^2(x,t)\,\mathrm{d}x,$$

and

$$\begin{aligned} I''(t) &= 2\int_\Omega u u_t\,\mathrm{d}x \\ &= 2\int_{\partial\Omega} |u|^p \mathrm{d}\sigma - 2\Big(1 - \int_0^t g(s)\mathrm{d}s\Big)\int_\Omega |\nabla u|^2 \mathrm{d}x \\ &\quad + 2\int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(s) - \nabla u(t))\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{2.7}$$

By using Cauchy–Schwarz's inequality first and Young's inequality next, we get from (2.7) that

$$\begin{aligned} I''(t) &\ge 2\int_{\partial\Omega} |u|^p \mathrm{d}\sigma - 2\Big(1 - \int_0^t g(s)\,\mathrm{d}s\Big)\int_\Omega |\nabla u|^2 \mathrm{d}x \\ &\quad - 2\int_0^t g(t-s)\|\nabla u(t)\|_2 \|\nabla u(t) - \nabla u(s)\|_2\,\mathrm{d}s \\ &\ge 2\int_{\partial\Omega} |u|^p\,\mathrm{d}\sigma - 2\Big(1 - \frac{3}{4}\int_0^t g(s)\,\mathrm{d}s\Big)\int_\Omega |\nabla u|^2 \mathrm{d}x - 2(g \diamond \nabla u)(t). \end{aligned} \tag{2.8}$$

By comparing the terms in (2.8) with (2.1) and noticing (2.2) and (2.5), we have, for some $\delta > 0$, that

$$\begin{aligned} I''(t) &\ge -4(1+\delta)E(t) \\ &= 4(1+\delta)\Big(-E(0) + \int_0^t \int_\Omega u_\tau^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_0^t (g(\tau)\int_\Omega |\nabla u(\tau)|^2 \mathrm{d}x)\,\mathrm{d}\tau \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int_0^t \int_0^\tau g'(\tau - s) \int_\Omega |\nabla u(s) - \nabla u(\tau)|^2 \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau\Big) \\
&\geq 4(1+\delta)\Big(- E(0) + \int_0^t \int_\Omega u_\tau^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big).
\end{aligned}
$$

Clearly,

$$
I'(t) = \int_\Omega u^2(x,t)\,\mathrm{d}x = 2\int_0^t \int_\Omega u u_\tau \,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega u_0^2(x)\,\mathrm{d}x. \tag{2.9}
$$

It follows that, for any $\varepsilon > 0$,

$$
I'(t)^2 \leq 4(1+\varepsilon)\int_0^t \int_\Omega u^2 \,\mathrm{d}x\,\mathrm{d}\tau \int_0^t \int_\Omega u_\tau^2 \,\mathrm{d}x\,\mathrm{d}\tau + (1+\frac{1}{\varepsilon})\Big(\int_\Omega u_0^2(x)\,\mathrm{d}x\Big)^2. \tag{2.10}
$$

Combining the above estimates, we find that for $\alpha > 0$,

$$
\begin{aligned}
&I''(t)I(t) - (1+\alpha)I'(t)^2 \\
&\quad \geq 4(1+\delta)\Big(- E(0) + \int_0^t \int_\Omega u_\tau^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big)\Big(\int_0^t \int_\Omega u^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + A\Big) \\
&\qquad - 4(1+\alpha)(1+\varepsilon)\int_0^t \int_\Omega u^2 \,\mathrm{d}x\,\mathrm{d}\tau \int_0^t \int_\Omega u_\tau^2 \,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad - (1+\alpha)(1+\frac{1}{\varepsilon})\Big(\int_\Omega u_0^2(x)\,\mathrm{d}x\Big)^2.
\end{aligned} \tag{2.11}
$$

Now we choose ε and α small enough such that

$$
1+\delta > (1+\alpha)(1+\varepsilon). \tag{2.12}
$$

Thus, it is clear from (2.11), (2.12) and the assumption $E(0) < 0$ that

$$
I''(t)I(t) - (1+\alpha)I'(t)^2 > 0, \tag{2.13}
$$

for $A > 0$ large enough, which implies that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{I'(t)}{I^{1+\alpha}(t)}\Big) > 0,
$$

or

$$
\frac{I'(t)}{I^{1+\alpha}(t)} \geq \frac{I'(0)}{I^{1+\alpha}(0)},
$$

as long as u exists. It follows from the above inequality that $I(t) = \int_0^t \int_\Omega u^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + A$ cannot remain finite for all $t > 0$. The proof is complete. □

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