Partial differential equations

# A note on the variational analysis of the parabolic–parabolic Keller–Segel system in one spatial dimension

## Une note sur l'analyse variationelle du système de Keller–Segel parabolique–parabolique à une dimension spatiale

### Jonathan Zinsl

*Zentrum für Mathematik, Technische Universität München, 85747 Garching, Germany*

A B S T R A C T

We prove the existence of global-in-time weak solutions to a version of the parabolic–parabolic Keller–Segel system in one spatial dimension. If the coupling of the system is suitably weak, we prove the convergence of those solutions to the unique equilibrium with an exponential rate. Our proofs are based on an underlying gradient flow structure with respect to a mixed Wasserstein-$L^2$ distance.

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R É S U M É

Nous prouvons l'existence de solutions faibles globales en temps d'une variante du système de Keller–Segel parabolique–parabolique à une dimension spatiale. Si le couplage du système est assez faible, nous prouvons la convergence de ces solutions vers l'équilibre univoque à une vitesse exponentielle. Nos preuves reposent sur une structure de flux de gradient dans l'espace produit des espaces Wasserstein et $L^2$.

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## 1. Introduction and main results

We consider the following version of the Keller–Segel model for chemotaxis in one spatial dimension:

$$u_t(t,x) = (u_x(t,x) + u(t,x)W_x(x) - \chi u(t,x)v_x(t,x))_x, \tag{1}$$

$$v_t(t,x) = v_{xx}(t,x) - \kappa v(t,x) + \chi u(t,x), \tag{2}$$

where $t > 0$ and $x \in \mathbb{R}$, and the sought solution $(u, v)$ is subject to the initial conditions:

$$u(0,x) = u^0(x), \qquad v(0,x) = v^0(x). \tag{3}$$

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We require that $\kappa \geq 0$, $\chi \in \mathbb{R}$ and the *confinement potential* $W \in C^2(\mathbb{R})$ is bounded from below, grows at most quadratically, i.e. $\underline{W} \leq W(x) \leq Ax^2 + B$ for all $x \in \mathbb{R}$ and some $\underline{W}, A, B \in \mathbb{R}$, and has a bounded second derivative $W_{xx}$. It is known that (1)–(2) possess a variational structure since they can formally be written as gradient flows of the (non-convex) free-energy functional $\mathcal{H} : \mathbf{X} \to \mathbb{R} \cup \{\infty\}$ (see formula (4)) with respect to the compound distance $\mathbf{dist}((u, v), (u', v')) := \sqrt{\mathbf{d}_{\mathbf{W}_2}^2(u, u') + \|v - v'\|_{L^2}^2}$ on the space $\mathbf{X} := \mathscr{P}_2(\mathbb{R}) \times L^2(\mathbb{R})$, where $(\mathscr{P}_2(\mathbb{R}), \mathbf{d}_{\mathbf{W}_2})$ is the space of (absolutely continuous) probability measures—or their densities, respectively—on $\mathbb{R}$ with finite second moment $\mathbf{m}_2$, endowed with the $L^2$-Wasserstein distance $\mathbf{d}_{\mathbf{W}_2}$. The energy $\mathcal{H}$ is defined as

$$\mathcal{H}(u, v) = \begin{cases} \int_{\mathbb{R}} \left[ u \log u + uW + \frac{1}{2}v_x^2 + \frac{\kappa}{2}v^2 - \chi uv \right] \mathrm{d}x, & \text{if } \int_{\mathbb{R}} u \log u \, \mathrm{d}x < \infty \text{ and } v \in H^1(\mathbb{R}), \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

In system (1)–(2), the chemotactic sensitivity $\chi$ necessarily coincides with the production rate for the chemoattractant in order to obtain a formal gradient structure. Following the procedure in [4,16], the results presented here hold with minor changes in the proofs also for more general systems allowing, e.g., different values for those two parameters. Furthermore, owing to the fact that equation (1) formally conserves mass, a renormalisation of the $u$ component to unit mass is possible (see for instance [4] for more details). A smallness condition for $\chi$ (as, e.g., required in Theorem 1.2) corresponds to a smallness condition on the mass of the initial density in the non-renormalised system.

In this note, we sketch another application of the method in [17] to prove the existence of weak solutions to (1)–(2) and to analyse their long-time behaviour. There, the global-in-time existence of weak solutions and their exponential convergence to the unique equilibrium in the regime of small coupling have been shown in the case of a porous-medium-type diffusion for $u$ on $\mathbb{R}^3$. In the one-dimensional setting, the proof is considerably simpler compared to [17] due to a gain in regularity. In contrast, the case of linear diffusion causes the difficulty of a missing time-uniform *a priori* estimate for $u$ in $L^m(\mathbb{R})$ for some $m > 1$.

The cornerstone of our variational analysis is the so-called *minimizing movement scheme* (see, e.g., [1,9]) for the construction of an approximate time-discrete solution: for each step size $\tau > 0$, let $(u_\tau^0, v_\tau^0) := (u^0, v^0)$, and then define inductively for each $n \in \mathbb{N}$:

$$(u_\tau^n, v_\tau^n) \in \operatorname*{argmin}_{(u,v) \in \mathscr{P}_2(\mathbb{R}) \times L^2(\mathbb{R})} \left( \frac{1}{2\tau} \mathbf{dist}\left((u, v), (u_\tau^{n-1}, v_\tau^{n-1})\right)^2 + \mathcal{H}(u, v) \right). \tag{5}$$

Further, introduce the piecewise constant interpolation $(u_\tau, v_\tau) : \mathbb{R}_+ \to \mathscr{P}_2(\mathbb{R}) \times L^2(\mathbb{R})$ by

$$u_\tau(t) = u_\tau^n, \quad v_\tau(t) = v_\tau^n \quad \text{for all } t \in ((n-1)\tau, n\tau]. \tag{6}$$

This hybrid variational principle has been exploited previously for Keller–Segel-type systems [5,4,14,16] in higher spatial dimensions and also in other applications, e.g., [10,12]. For the vast literature on the behaviour of the Keller–Segel system and its variants, we refer the reader to the review articles by Horstmann [8] and Blanchet [3], and emphasise that the one-dimensional model on bounded spatial domains has been explicitly investigated by Osaki and Yagi [15] and Hillen and Potapov [7], leading to similar results.

We obtain the following on the existence of global-in-time weak solutions.

**Theorem 1.1** *(Existence). Assume that* $\chi$*,* $\kappa$ *and* $W$ *are as mentioned above and that the initial condition satisfies* $u^0 \in \mathscr{P}_2(\mathbb{R})$*,* $\int_{\mathbb{R}} u^0 \log u^0 \, \mathrm{d}x < \infty$ *and* $v^0 \in H^1(\mathbb{R})$*. Define, for each* $\tau > 0$*, a discrete solution by* (5)–(6)*. Then, there exists a sequence* $\tau_k \searrow 0$ $(k \to \infty)$ *such that* $(u_{\tau_k}, v_{\tau_k})$ *converges to a weak solution* $(u, v)$ *to* (1)–(3) *in the sense that* (1) *holds in the sense of distributions, whereas* (2) *and* (3) *hold almost everywhere. One has for all* $T > 0$*:*

$u_{\tau_k} \rightharpoonup u$ *narrowly in the space of probability measures* $\mathscr{P}(\mathbb{R})$*, pointwise with respect to* $t \in [0, T]$*,*

$v_{\tau_k} \to v$ *in* $L^2(\mathbb{R})$*, uniformly with respect to* $t \in [0, T]$*,*

$u \in C^{1/2}([0, T]; (\mathscr{P}_2(\mathbb{R}), \mathbf{d}_{\mathbf{W}_2})) \cap L^1([0, T]; L^\infty(\mathbb{R})) \cap L^2([0, T]; L^2(\mathbb{R}))$*,*

$\sqrt{u} \in L^2([0, T]; H^1(\mathbb{R}))$*,* $u \log u \in L^\infty([0, T]; L^1(\mathbb{R}))$*,*

$v \in C^0([0, T] \times \mathbb{R}) \cap H^1([0, T]; L^2(\mathbb{R})) \cap L^\infty([0, T]; H^1(\mathbb{R})) \cap L^2([0, T]; H^2(\mathbb{R}))$*.*

In particular, for fixed $t > 0$, $u(t, \cdot)$ is nonnegative, continuous and bounded. The second component $v$ is bounded and continuous even in *both* variables. At least in the case $\chi \geq 0$, its nonnegativity can be obtained starting with a nonnegative initial condition.

Our result on the long-time behaviour of the weak solution from Theorem 1.1 reads as follows.

**Theorem 1.2** *(Convergence to equilibrium). Assume in addition to the hypotheses of Theorem 1.1 that* $W$ *is* $\lambda_0$*-convex for some* $\lambda_0 > 0$ *and that* $\kappa > 0$ *is strictly positive. There exist* $\bar\varepsilon > 0$*,* $C > 0$ *and* $L > 0$ *such that for all* $\chi = \varepsilon \in (0, \bar\varepsilon)$*, the following statements hold:*

(a) *system* (1)–(2) *possesses a unique stationary state* $(u^\infty, v^\infty) \in (\mathscr{P}_2 \cap L^\infty)(\mathbb{R}) \times H^2(\mathbb{R})$ *satisfying*

$$v_{xx}^\infty = \kappa v^\infty - \varepsilon \left[ \int_\mathbb{R} \exp(-W + \varepsilon v^\infty) \, \mathrm{d}x \right]^{-1} \exp(-W + \varepsilon v^\infty),$$

$$u^\infty = \left[ \int_\mathbb{R} \exp(-W + \varepsilon v^\infty) \, \mathrm{d}x \right]^{-1} \exp(-W + \varepsilon v^\infty);$$

(b) *with* $\Lambda_\varepsilon := \min(\kappa, \lambda_0) - \varepsilon L > 0$, *the weak solution* $(u, v)$ *to* (1)–(3) *from Theorem 1.1 admits for all* $t \geq 0$ *the estimate*

$$\|u(t) - u^\infty\|_{L^1} + \mathbf{d}_{\mathbf{W}_2}(u(t), u^\infty) + \sup_{x \in \mathbb{R}} |v(t) - v^\infty| + \|v(t) - v^\infty\|_{H^1}$$
$$\leq C(\mathcal{H}(u^0, v^0) - \mathcal{H}(u^\infty, v^\infty))^{1/2} \, \mathrm{e}^{-\Lambda_\varepsilon t}, \tag{7}$$

*i.e.* $(u(t), v(t))$ *converges exponentially fast with rate* $\Lambda_\varepsilon$ *to the equilibrium* $(u^\infty, v^\infty)$ *as* $t \to \infty$.

The resulting convergence estimate (7) is a consequence of specific Sobolev embeddings that hold in one dimension only. A weaker result of the same kind has been proven in [17] in space dimension three with more technical effort. In contrast, the question of long-time behaviour of weak solutions to the parabolic–parabolic Keller–Segel system *on the plane* remains open. The right estimates for applying the strategy presented here do not seem to be at hand easily.

## 2. Sketch of proof for Theorem 1.1

The crucial step in the proof of Theorem 1.1 is to verify that the discrete solution $(u_\tau, v_\tau)$ is well defined and regular enough to allow for the passage to the continuous-time limit $\tau \searrow 0$ in a strong sense. Once obtained, we can proceed as in [16,17], establishing an approximate weak formulation that turns into the weak formulation of the time-continuous equation as $\tau \searrow 0$. We prove the following.

**Proposition 2.1** *(Discrete solution). For each* $\tau > 0$ *and* $(\tilde{u}, \tilde{v}) \in \mathbf{X}$, *the functional* $\mathcal{H}_\tau(\cdot | \tilde{u}, \tilde{v}) := \frac{1}{2\tau} \mathbf{dist}^2(\cdot, (\tilde{u}, \tilde{v})) + \mathcal{H}$ *possesses a minimizer* $(u, v) \in \mathscr{P}_2(\mathbb{R}) \times H^1(\mathbb{R})$ *with* $\int_\mathbb{R} u \log u \, \mathrm{d}x < \infty$. *Moreover, there exist constants* $K_0, K_1, K_2 > 0$ *such that*

$$\tau \|(\sqrt{u})_x\|_{L^2}^2 + \tau \|v_{xx}\|_{L^2}^2 \leq K_0 \int_\mathbb{R} (u \log u - \tilde{u} \log \tilde{u}) \, \mathrm{d}x + K_1 (\|v\|_{H^1}^2 - \|\tilde{v}\|_{H^1}^2) + K_2 \tau (\|v\|_{H^1}^2 + 1). \tag{8}$$

*If additionally* $\tilde{v} \in H^1(\mathbb{R})$ *and* $\int_\mathbb{R} \tilde{u} \log \tilde{u} \, \mathrm{d}x < \infty$, *then* $v \in H^2(\mathbb{R})$, $\sqrt{u} \in H^1(\mathbb{R})$ *and* $u \in L^\infty(\mathbb{R})$.

**Proof.** First, in one spatial dimension, there exists $C_0 > 0$ such that $\|v\|_{L^\infty} \leq \|v\|_{C^{0,\frac{1}{2}}} \leq C_0 \|v\|_{H^1}$. Moreover, for some $C_1 > 0$, one has

$$\int_\mathbb{R} u \log u \, \mathrm{d}x \geq -C_1 (\mathbf{m}_2(u) + 1)^{1/2}.$$

From this, we easily see that for all $(u, v) \in \mathscr{P}_2(\mathbb{R}) \times H^1(\mathbb{R})$ with $\int_\mathbb{R} u \log u \, \mathrm{d}x < \infty$, we have

$$\int_\mathbb{R} u \log u \, \mathrm{d}x + \underline{W} + \frac{1}{2} \|v_x\|_{L^2}^2 - |\chi| C_0 \|v\|_{H^1} \leq \mathcal{H}(u, v) < \infty.$$

Using the triangle inequality for **dist** and Young's inequality, we deduce coercivity of $\mathcal{H}_\tau(\cdot | \tilde{u}, \tilde{v})$:

$$\mathcal{H}_\tau(u, v | \tilde{u}, \tilde{v}) \geq \frac{1}{4} \|v\|_{H^1}^2 + \frac{1}{4} \mathbf{m}_2(u) - C.$$

Thus, by the Banach–Alaoglu, Arzelà–Ascoli and Prokhorov theorems, a minimizing sequence $(u_n, v_n)_{n \in \mathbb{N}}$ for $\mathcal{H}_\tau(\cdot | \tilde{u}, \tilde{v})$ converges—at least on a subsequence—to some limit $(u, v) \in \mathscr{P}_2(\mathbb{R}) \times H^1(\mathbb{R})$ with $\int_\mathbb{R} u \log u \, \mathrm{d}x < \infty$: $v_n \rightharpoonup v$ in $H^1(\mathbb{R})$, $v_n \to v$ locally uniformly in $\mathbb{R}$ and $u_n \rightharpoonup u$ narrowly in $\mathscr{P}(\mathbb{R})$. With respect to these convergences, $\mathcal{H}_\tau(\cdot | \tilde{u}, \tilde{v})$ is lower semicontinuous, which is clear except for the term $\int_\mathbb{R} u_n v_n \, \mathrm{d}x$. We employ a truncation argument similar to that in [16] to prove l.s.c. for this remaining term, and consequently obtain the minimizing property for $(u, v)$. It remains to prove the additional regularity estimate (8). We investigate the dissipation of $\mathcal{H}$ along the (auxiliary) 0-flow $(\mathcal{U}^s, \mathcal{V}^s)_{s \geq 0}$ w.r.t. **dist** generated by the 0-geodesically convex functional

$$\mathcal{E}(u, v) := \int_{\mathbb{R}} \left[ u \log u + \frac{1}{2} v_x^2 + \frac{\kappa}{2} v^2 \right] dx$$

on **X**. Elementary calculations yield, since we have $\mathcal{U}_s^s = \mathcal{U}_{xx}^s$, $\mathcal{V}_s^s = \mathcal{V}_{xx}^s - \kappa \mathcal{V}^s$:

$$\frac{d}{ds} \mathcal{H}(\mathcal{U}^s(u), \mathcal{V}^s(v)) \leq \int_{\mathbb{R}} \left[ -4(\sqrt{\mathcal{U}^s})_x^2 + \|W_{xx}\|_{L^\infty} - \frac{1}{2}(\mathcal{V}_{xx}^s - \kappa \mathcal{V}^s)^2 + \frac{5}{2}\chi^2(\mathcal{U}^s)^2 + \frac{\kappa^2}{2}(\mathcal{V}^s)^2 \right] dx.$$

Using the Sobolev inequality $\|\eta\|_{L^4} \leq C \|\eta\|_{H^1}^{1/4} \|\eta\|_{L^2}^{3/4}$, we eventually arrive at

$$\frac{d}{ds} \mathcal{H}(\mathcal{U}^s(u), \mathcal{V}^s(v)) \leq -2\|(\sqrt{\mathcal{U}^s})_x\|_{L^2}^2 - \frac{1}{2}\|\mathcal{V}_{xx}^s - \kappa \mathcal{V}^s\|_{L^2}^2 + \frac{\kappa^2}{2}\|v\|_{L^2}^2 + C_2. \tag{9}$$

Finally, we use the *flow interchange lemma* [13, Thm. 3.2] to obtain $\mathcal{E}(u, v) + \tau D^{\mathcal{E}} \mathcal{H}(u, v) \leq \mathcal{E}(\tilde{u}, \tilde{v})$, which yields (8) in combination with (9) and lower semicontinuity as $s \searrow 0$. $\square$

Proceeding as in [16,17], we end up with a weak solution $(u, v)$ to (1)–(3) with the properties

$$v \in L^\infty([0, T]; L^2(\mathbb{R})), \ v_x \in L^\infty([0, T]; L^2(\mathbb{R})), \ v_t \in L^2([0, T]; L^2(\mathbb{R})).$$

We immediately deduce that $v \in L^\infty([0, T] \times \mathbb{R})$. We now show that $v$ is continuous in both arguments. In fact, for all bounded intervals $I \subset \mathbb{R}$, $v$ belongs to the *anisotropic* Sobolev space $W^{1,\mathbf{P}}([0, T] \times I)$ with $\mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, the spectral radius of which is less than 1. Since in this case $W^{1,\mathbf{P}}([0, T] \times I) \Subset C^0([0, T] \times \bar{I})$, the claim follows (for details on anisotropic spaces, see, e.g., [2,11]).

## 3. Sketch of proof for Theorem 1.2

The additional assumption of $\lambda_0$-convexity of the confinement $W$ yields boundedness from below of the energy $\mathcal{H}$. We obtain $(u^\infty, v^\infty) \in (\mathscr{P}_2 \cap L^\infty)(\mathbb{R}) \times H^2(\mathbb{R})$ as the unique minimizer of $\mathcal{H}$ in a fashion similar to that in [17], using the strict convexity of $\mathcal{H}$ on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ for small coupling strength $\varepsilon > 0$.

We observe that the energy can be decomposed as follows into a convex part $\mathcal{L}$ (see Proposition 3.1) and a non-convex, but controllable part $\varepsilon \mathcal{L}_*$:

$$\mathcal{H}(u, v) - \mathcal{H}(u^\infty, v^\infty) = \mathcal{L}(u, v) + \varepsilon \mathcal{L}_*(u, v), \tag{10}$$

where $\mathcal{L}(u, v) = \mathcal{L}_u(u) + \mathcal{L}_v(v)$,

$$\mathcal{L}_u(u) := \int_{\mathbb{R}} \left[ u \log u - u^\infty \log u^\infty + W^\varepsilon (u - u^\infty) \right] dx, \ \text{with } W^\varepsilon := W - \varepsilon v^\infty,$$

$$\mathcal{L}_v(v) := \frac{1}{2} \|(v - v^\infty)_x\|_{L^2}^2 + \frac{\kappa}{2} \|v - v^\infty\|_{L^2}^2, \quad \mathcal{L}_*(u, v) := -\int_{\mathbb{R}} (u - u^\infty)(v - v^\infty) \, dx.$$

**Proposition 3.1** (*Properties of $\mathcal{L}$*). *Let $\varepsilon$ be sufficiently small. Then, the following statements hold:*

(a) *there exists $M_1 > 0$ such that the* perturbed potential $W^\varepsilon$ *is $\lambda_\varepsilon$-convex, where $\lambda_\varepsilon := \lambda_0 - M_1 \varepsilon > 0$;*
(b) *the functional $\mathcal{L}_u$ is $\lambda_\varepsilon$-geodesically convex on $(\mathscr{P}_2(\mathbb{R}), \mathbf{d}_{\mathbf{W}_2})$ and*

$$\frac{\lambda_\varepsilon}{2} \mathbf{d}_{\mathbf{W}_2}^2(u, u^\infty) \leq \mathcal{L}_u(u) \leq \frac{1}{2\lambda_\varepsilon} \int_{\mathbb{R}} u((\log u + W^\varepsilon)_x)^2 \, dx;$$

(c) *the functional $\mathcal{L}_v$ is $\kappa$-geodesically convex on $L^2(\mathbb{R})$ and*

$$\frac{\kappa}{2} \|v - v^\infty\|_{L^2}^2 \leq \mathcal{L}_v(v) \leq \frac{1}{2\kappa} \|(v - v^\infty)_{xx} - \kappa(v - v^\infty)\|_{L^2}^2;$$

(d) *there exists $M_2 > 0$ such that $\mathcal{L}(u, v) \leq (1 + M_2 \varepsilon)(\mathcal{H}(u, v) - \mathcal{H}(u^\infty, v^\infty))$;*
(e) *there exists $C' > 0$ such that*

$$\|u - u^\infty\|_{L^2}^2 \leq C' \int_{\mathbb{R}} u((\log u + W^\varepsilon)_x)^2 \, dx. \tag{11}$$

Actually, in one spatial dimension, the proof of part (a) simplifies dramatically compared to [17], since

$$W^\varepsilon_{xx} = W_{xx} - \varepsilon v^\infty_{xx} = W_{xx} - \varepsilon(\kappa v^\infty - \varepsilon u^\infty) \geq \lambda_0 - \varepsilon\kappa\|v^\infty\|_{L^\infty} \geq \lambda_0 - \varepsilon\tilde{C}(\mathcal{H}(u^\infty, v^\infty) + 1),$$

for some constant $\tilde{C} > 0$. The proof of part (d) is mainly a consequence of the *Csiszár–Kullback inequality* (cf. [6]) $\|u - u^\infty\|^2_{L^1} \leq C\mathcal{L}_u(u)$. The idea of proof of (e) is as follows: we distinguish the cases where the integral on the r.h.s. in (11) is small or large, respectively. In the former case, we can deduce from that an $L^\infty$ bound on $u$ leading to the desired result using the Taylor expansion of the integrand in $\mathcal{L}_u$ at $u_\infty(x)$. The latter case can be treated by a suitable Sobolev interpolation in one spatial dimension.

We now prove the central estimate leading to Theorem 1.2:

**Proposition 3.2** *(Exponential estimate for $\mathcal{L}$). Let $(u^n_\tau, v^n_\tau)_{n\in\mathbb{N}}$ be a family of time-discrete approximations obtained by* (5) *that converges to a weak solution $(u, v)$ as $\tau \searrow 0$ in the sense stated in Theorem 1.1. Then, there exist $\bar{\varepsilon} > 0$ and $L > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ and $n \in \mathbb{N}$, one has*

$$\mathcal{L}(u^n_\tau, v^n_\tau) \leq (1 + M_2\varepsilon)(\mathcal{H}(u^0, v^0) - \mathcal{H}(u^\infty, v^\infty))(1 + 2\Lambda_\varepsilon\tau)^{-n}, \tag{12}$$

*with $\Lambda_\varepsilon := \min(\lambda_0, \kappa) - L\varepsilon > 0$.*

Once proven, this result yields the exponential convergence of $\mathcal{L}(u(t), v(t))$ to zero for $t \to \infty$ after passage to the continuous-time limit $\tau \searrow 0$. From this, Theorem 1.2 clearly follows.

**Proof.** We investigate the dissipation of $\mathcal{H}$ along the (auxiliary) $\min(\lambda_\varepsilon, \kappa)$-flow $(\mathcal{U}^s, \mathcal{V}^s)_{s\geq0}$ of the $\min(\lambda_\varepsilon, \kappa)$-geodesically convex functional $\mathcal{L}$ on $\mathbf{X}$, which is associated with the evolution system:

$$\mathcal{U}^s_s = (\mathcal{U}^s_x + \mathcal{U}^s W^\varepsilon_x)_x, \qquad \mathcal{V}^s_s = (\mathcal{V}^s - v^\infty)_{xx} - \kappa(\mathcal{V}^s - v^\infty).$$

First, by elementary calculations, we obtain, using decomposition (10), that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}(\mathcal{U}^s(u), \mathcal{V}^s(v)) \leq \left(\frac{\varepsilon}{2} - 1\right)\int_\mathbb{R} \mathcal{U}^s((\log\mathcal{U}^s + W^\varepsilon)_x)^2\,\mathrm{d}x + \frac{\varepsilon}{2}\int_\mathbb{R}\mathcal{U}^s(\mathcal{V}^s - v^\infty)^2_x\,\mathrm{d}x$$

$$+ \frac{\varepsilon}{2}\|\mathcal{U}^s - u^\infty\|^2_{L^2} + \left(\frac{\varepsilon}{2} - 1\right)\|(\mathcal{V}^s - v^\infty)_{xx} - \kappa(\mathcal{V}^s - v^\infty)\|^2_{L^2}.$$

The third term can be controlled by the first one using (11), whereas the second term is to be controlled by the fourth term using the inequality $\|\eta\mu^2_x\|_{L^1} \leq C\|\eta\|_{L^1}\|\mu\|^2_{H^2}$, which is valid in one spatial dimension. Taking into account the properties of $\mathcal{L}$ from Proposition 3.1, we end up with

$$-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}(\mathcal{U}^s(u), \mathcal{V}^s(v)) \geq 2(1 - \varepsilon M)\min(\lambda_\varepsilon, \kappa)\mathcal{L}(\mathcal{U}^s(u), \mathcal{V}^s(v)), \tag{13}$$

for some constant $M > 0$ if $\varepsilon$ is sufficiently small. The application of the flow interchange lemma [13, Thm. 3.2] eventually yields with (13): $[1 + 2\tau(1 - \varepsilon M)\min(\lambda_\varepsilon, \kappa)]\mathcal{L}(u^n_\tau, v^n_\tau) \leq \mathcal{L}(u^{n-1}_\tau, v^{n-1}_\tau)$. By iteration of this estimate and Proposition 3.1(d), the desired estimate (12) follows. □

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