Algebraic geometry

On a question of Mehta and Pauly

Sur une question de Mehta et Pauly

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\textbf{A R T I C L E   I N F O}

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\textbf{A B S T R A C T}

In this short note, we provide explicit examples in characteristic $p$ on certain smooth projective curves where for a given semistable vector bundle $\mathcal{E}$ the length of the Harder–Narasimhan filtration of $F^*\mathcal{E}$ is longer than $p$. This negatively answers a question of Mehta and Pauly raised in [2].

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\textbf{R É S U M É}

Dans cette courte note, nous donnons des exemples explicites en caractéristique $p$ sur certaines courbes projectives lisses où, pour un fibré vectoriel semi-stable donné $\mathcal{E}$, la longueur de la filtration de Harder–Narasimhan de $F^*\mathcal{E}$ est plus grande que $p$. Cela répond négativement à une question posée par Mehta et Pauly dans [2].

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0. Introduction

In [2, page 2], Mehta and Pauly asked whether for a smooth projective curve over a field of characteristic $p > 0$ and $\mathcal{E}$ a semistable bundle on $X$ the length of the Harder–Narasimhan filtration of $F^*\mathcal{E}$ is at most $p$. In [4, Construction 2.13], this is answered negatively. Examples are constructed based on a result of Sun [3]. The bundles for which examples are obtained in [4] have rank $\geq 2p$ (in fact, examples are constructed for any $np$ with $n \geq 2$) and are over curves of large genus, since restriction theorems and Bertini’s Theorem are used. The purpose of this short note is to provide surprisingly simple down-to-earth examples in characteristic $p$ for certain smooth plane curves and bundles of rank $p + 1 \leq r \leq \left\lfloor \frac{3p+1}{2} \right\rfloor$. In characteristic 2, negative examples exist on any smooth projective curve of genus $\geq 2$. We note that our examples are only polystable, while one should be able to obtain stable bundles using the methods outlined in [4].

1. The example

\textbf{Proposition 1.1.} Let $X$ be a smooth projective curve over an algebraically closed field $k$ of positive characteristic. Let $\mathcal{E}_i$, $i = 1, \ldots, n$ be semistable rank-two bundles of slope $\mu$ on $X$ such that the $F^*\mathcal{E}_i$ split as $F^*\mathcal{E}_i = \mathcal{L}_i \oplus \mathcal{G}_i$ with $\mu(\mathcal{L}_i) > \mu(\mathcal{G}_i)$. Assume, moreover, that $\ldots$
\( \mu(L_i) > \mu(L_{i+1}) \) for all \( i = 1, \ldots, n-1 \). Then \( S = \bigoplus_{i=1}^{n} E_i \) is semistable and \( F^*S \) is unstable and its Harder–Narasimhan filtration is:

\[
0 \subset L_1 \subset L_1 \oplus L_2 \subset \cdots \subset \bigoplus_{i=1}^{n} L_i \subset \bigoplus_{i=1}^{n} L_i \oplus G_n \subset \bigoplus_{i=1}^{n} L_i \oplus G_n \oplus G_{n-1} \subset \cdots \subset F^*S.
\]

In particular, the Harder–Narasimhan filtration of \( F^*S \) has length \( 2n \).

**Proof.** Clearly \( S \) is semistable. We have \( \mu(G_i) = 2\mu - \mu(L_i) \), which implies \( \mu(G_i) < \mu(G_{i+1}) \) for all \( i \). We also have \( \mu(L_i) > \mu(L_{i+1}) \) for all \( i, j \). Indeed, we may assume that \( i > j \) then \( \mu(L_i) - \mu(L_j) = \mu(L_j) - \mu(G_j) \) and by assumption \( \mu(L_i) > \mu(L_j) > \mu(G_j) \). Hence, \( \mu(L_i) > \mu(G_j) \).

It follows that the slopes of the quotients \( Q_i \) of the filtration form a strictly decreasing sequence. As all \( Q_i \) are semistable as line bundles, this is the Harder–Narasimhan filtration of \( F^*S \). \( \square \)

**Example 1.2.** By [1, Theorem 1] any smooth projective curve \( X \) of genus \( \geq 2 \) admits a semistable rank two bundle \( E \) with trivial determinant such that \( F^*E \) is not semistable. Then \( S = E \oplus O_X \) is a semistable vector bundle and the Harder–Narasimhan filtration of \( F^*S \) has length \( 3 > 2 \). Indeed, if \( 0 \subset L \subset F^*E \) is a Harder–Narasimhan filtration of \( F^*E \) then \( 0 \subset L \subset L \oplus O_X \subset F^*S \) is one for \( F^*S \).

**Lemma 1.3.** Let \( X \) be a smooth projective curve and \( E \) a rank 2 vector bundle on \( X \). If \( E \) is given by an extension \( 0 \neq c \in \text{Ext}^1(M, L) \) with \( \text{deg} L = \text{deg} M \) and \( F^*(c) = 0 \) then \( E \) is semistable.

**Proof.** Assume, on the contrary, that \( E \) is unstable and let \( \mathcal{N} \) denote the maximal destabilizing subbundle \( \mathcal{E} \). The maximal destabilizing subbundle of \( F^*E = F^*M \oplus F^*L \) is \( F^*\mathcal{N} \). Since the Harder–Narasimhan filtration is unique and in the rank 2 case automatically strong, we must have \( F^*M = F^*\mathcal{N} \). Hence, \( \mathcal{N} = M \oplus T \) for some \( p \)-torsion bundle \( T \).

Consider now the natural inclusion \( i : M \otimes T \to E \) and the projection \( p : E \to M \). The Frobenius pull-back of the composition \( p \circ i \) is the identity. In particular \( p \circ i : M \otimes T \to M \) is non-zero. Since both line bundles are of the same degree, this map is an isomorphism. Hence, if \( E \) is not semistable, then the sequence has to split, which contradicts the assumption \( c \neq 0 \). \( \square \)

**Example 1.4.** Let now \( p \) be any prime and \( k \) an algebraically closed field of characteristic \( p \). We consider the plane curve:

\[
X = V_+(x^{3p} + xy^{3p-1} + y^{3p-1}) \subseteq \mathbb{P}^2_k.
\]

By the Jacobian criterion, this is a smooth curve. We will construct \( \left\lfloor \frac{3p+1}{2} \right\rfloor \) rank-two bundles of slopes \( -\frac{3p}{2} \) as in Proposition 1.1. The direct sum over at least \( \frac{p+1}{2} \) of these bundles then constitutes the desired example.

Consider the cohomology class

\[
c = \frac{x^3}{yz^2} \in H^1(X, O_X(-1)),
\]

which is non-zero. Also note that its Frobenius pull-back

\[
F^*(c) = \frac{x^{2p}}{y^{2p} z^{2p}} = -\frac{xy^{3p-1} - yz^{3p-1}}{y^{2p} z^{2p}} = -\frac{xy^{p-1}}{z^{2p}} + \frac{z^{p-1}}{y^{2p-1}}
\]

is zero. Moreover, multiplication by \( z \) yields a map \( O_X(-1) \to O_X \) and the induced map on cohomology maps \( c \) to \( \frac{x^d}{y^{2p}} \), which is still non-zero. Let \( P_1, \ldots, P_{3p} \) be the (distinct) points on \( X \) where \( z \) vanishes.\(^2\) In particular, the cokernel of multiplication by \( z \) is just \( \bigoplus_{i=1}^{3p} k(P_i) \), where \( k(P_i) \) is the skyscraper sheaf at \( P_i \).

Multiplication by \( z \) factors as

\[
O_X(-1) \longrightarrow O_X(-1 + \sum_{i=1}^{p} P_i) \longrightarrow \bigoplus_{i=1}^{p} k(P_i)
\]

for any \( l \leq 3p \). Indeed, the image of the line bundle in the middle is just the sum of the image of \( O_X(-1) \) in \( O_X \) and the preimage of \( \sum_{i=1}^{p} k(P_i) \). In particular, we get an induced factorization on cohomology and we denote the image of \( c \) in \( H^1(X, O_X(-1 + \sum_{i=1}^{p} P_i)) \) by \( c_i \). Note that \( c_i \) is non-zero, while \( F^*(c_i) \) is zero.

Assume now that \( l \) is even. These cohomology classes then define extensions \( E_i \) as follows. Let \( l \) be the odd numbers from \( 1 \) to \( l \) and let \( j \) be the even numbers from \( 1 \) to \( l \). Then

\(^2\) We could also work with multiplication by \( x \) which yields one reduced point and one with multiplicity \( 3p - 1 \).
cl ∈ H^1(X, O_X(−1 + \sum_{i=1}^l k(P_i))) = \text{Ext}^1(O_X(−\sum_{j\in J} P_j), O_X(−1 + \sum_{i\in I} P_i))

yield extensions

0 \longrightarrow O_X(−1 + \sum_{i\in I} P_i) \longrightarrow \mathcal{E}_l \longrightarrow O_X(−\sum_{j\in J} P_j) \longrightarrow 0.

The \mathcal{E}_l all have slope \(-\frac{3p}{2}\) and pulling back along Frobenius splits the above sequence. By Lemma 1.3 the \mathcal{E}_l are semistable. Hence, the \mathcal{E}_l satisfy the hypothesis of Proposition 1.1, and we obtain the desired examples.

References