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Harmonic analysis

On irregular sampling in Bernstein spaces

Sur l'échantillonnage irrégulier dans les espaces de Bernstein

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A R T I C L E I N F O

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A B S T R A C T

We obtain sharp estimates for the sampling constants in Bernstein spaces when the density of the sampling set is near the critical value.

R É S U M É

Nous obtenons des estimations finales pour les constantes de l'échantillonnage dans les espaces de Bernstein lorsque la densité des échantillons est proche de la valeur critique.

1. Introduction

Given a number $\sigma > 0$, the Bernstein space B_σ is defined to be the set of all entire functions f satisfying for all real x and y the inequality $|f(x+iy)| \le C\exp(\sigma|y|)$ with some $C = C(f)$.

A set $\Lambda \subset \mathbb{R}$ is called uniformly discrete (u.d.) if

$$\inf_{\lambda,\lambda'\in\Lambda,\lambda\neq\lambda'} \left|\lambda - \lambda'\right| > 0.$$

One says that Λ is a (stable) sampling set for B_σ if there exists K such that

$$\|f\| := \sup_{t\in\mathbb{R}} \left|f(t)\right| \le K \sup_{\lambda\in\Lambda} \left|f(\lambda)\right| \quad (f \in B_\sigma).$$

The minimal constant K for which this holds is called the sampling constant $K(\Lambda, B_\sigma)$.

The classical Beurling theorem [2] characterizes sampling sets for B_σ in terms of the lower uniform density

$$D^-(\Lambda) := \lim_{l\to\infty} \min_{a\in\mathbb{R}} \frac{\#\Lambda \cap (a, a+l)}{l}.$$

Without loss of generality, one may consider the case $\sigma = \pi$. Then Beurling's theorem states:

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Λ is a sampling set for B_π if and only if $D^-(\Lambda) > 1$.

The most delicate point in Beurling's proof (see [2]) is to show that no sampling set Λ may have the critical density $D^-(\Lambda) = 1$.

If $D^-(\Lambda) = 1$, one can show that constant $K(\Lambda, B_\sigma)$ grows to infinity when σ approaches 1 from below. When $\Lambda = \mathbb{Z}$, S.N. Bernstein [1] proved that *the growth is precisely logarithmic:*

$$K(\mathbb{Z}, B_\sigma) = \frac{2}{\pi} \log \frac{\pi}{\pi - \sigma}\big(1 + o(1)\big) \quad (\sigma \uparrow \pi).$$

A slightly weaker result was proved in [3]. See also [6] where some estimates for $K(\Lambda, B_\sigma)$ are obtained. We mention also [4], where the Gabor frame considered for the Gaussian window, which corresponds to the lattice $a\mathbb{Z} \times a\mathbb{Z}$, and the asymptotics of the frames constants are obtained near the critical value $a = 1$.

2. Results

2.1. Sampling in Bernstein spaces

We are interested in the asymptotic behavior of the sampling constant $K(\Lambda, B_\sigma)$ for irregular sampling Λ near the critical value of density. Our main result shows that $K(\Lambda, B_\sigma)$ must have at least logarithmic growth.

We will denote by C different absolute positive constants.

Theorem 1. *Let Λ be a u.d. set with $D^-(\Lambda) = 1$. Then*

$$K(\Lambda, B_\sigma) \ge C \log \frac{\pi}{\pi - \sigma} \quad (0 < \sigma < \pi). \tag{1}$$

The proof is based on a reduction of the sampling problem to a similar one for the algebraic polynomials. This approach provides a new proof for the critical case in Beurling's theorem above.

It should be mentioned that removing even a single point from Λ may result in a much faster growth of the sampling constants. For example, it is straightforward to check that

$$K\big(\mathbb{Z} \setminus \{0\}, B_\sigma\big) \ge \frac{\sigma}{\pi - \sigma} \quad (0 < \sigma < \pi).$$

In fact, the constant $K(\Lambda, B_\sigma)$ may have arbitrarily fast growth:

Theorem 2. *For every function $\omega(\sigma) \uparrow \infty$ $(\sigma \uparrow \pi)$ there exists a u.d. set Λ, $D^-(\Lambda) = 1$, such that*

$$K(\Lambda, B_\sigma) \ge \omega(\sigma) \quad (\sigma < \pi).$$

2.2. Sampling in P_n

Denote by P_n the space of all algebraic polynomials of degree $\le n$ on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Given a finite set $\Lambda \subset \mathbb{T}$, $\#\Lambda > n$, one may introduce the corresponding sampling constant

$$K(\Lambda, P_n) := \sup_{P \in P_n, P \ne 0} \frac{\max_{z \in \mathbb{T}} |P(z)|}{\max_{\lambda \in \Lambda} |P(\lambda)|}.$$

Theorem 3. *For every $\Lambda \subset \mathbb{T}$, $\#\Lambda > n$, the estimate holds:*

$$K(\Lambda, P_n) \ge C \log \frac{n}{\#\Lambda - n}. \tag{2}$$

3. Sampling in spaces of polynomials

The following result essentially goes back to Faber:

Let U be a projector from the space $C(\mathbb{T})$ onto the subspace P_n. Then $\|U\| > C \log n$,

see [5], ch. 7.

Faber's approach is based on averaging over translations. Different versions of the result have been obtained by this approach. We will use the following one due to Al.A. Privalov [8] (see also [7]):

For every projector U above and every family of linear functionals ψ_j $(1 \le j \le m)$ in $C(\mathbb{T})$, there is a unit vector f in $C(\mathbb{T})$ such that $\|Uf\| > C \log n/m$, and the functionals vanish on f.

Proof of Theorem 3. Let n and m be positive integers. Given any $l := (n+1)+m$ points $\xi_j \in \mathbb{T}$ $(0 \le j \le l)$, for any $f \in C(\mathbb{T})$ denote by $P(f)$ the polynomial of degree n satisfying

$$P(f)(\xi_j) = f(\xi_j) \quad (0 \le j \le n).$$

Clearly, $P(f)$ is uniquely defined, and the operator $U : f \to P(f)$ is a projector from $C(\mathbb{T})$ onto P_n.

Set

$$\psi_j(f) := P(f)(\xi_{n+j}) \quad (1 \le j \le m).$$

Now we apply Privalov's theorem. We get a function f satisfying

$$\|f\| = 1, \qquad P(f)(\xi_j) = 0 \quad (n+1 \le j \le l), \qquad \|P(f)\| > C \log \frac{n}{m}.$$

Then (2) follows. □

4. Sampling in B_σ

We will sketch the proofs of Theorems 1 and 2. More details can be found in our preprint [7].

Let N be a positive integer and $\Lambda \subset [-N, N]$. Set

$$\Lambda_N := \Lambda \cup (-\infty, -N] \cup [N, \infty).$$

By Beurling's theorem, Λ_N is a sampling set for B_π. We show that for large N, the sampling constant $K(\Lambda_N, B_\pi)$ must be large unless the number of points of Λ in $(-N, N)$ is "much larger than" $2N$:

Proposition 1. *For every $\Lambda \subset [-N, N]$, $\#\Lambda > 2N$, we have:*

$$K(\Lambda_N, B_\pi) \ge C \log \frac{2N}{\#\Lambda - 2N}. \tag{3}$$

The proof consists of several steps.

1. First notice that by a simple change of variable in Theorem 3, one obtains:

Corollary 1. *Given $\nu \in \mathbb{N}$ and a set $\Gamma \subset [-\nu, \nu]$, $\#\Gamma > 2\nu$, there is an exponential polynomial*

$$P(t) = \sum_{|k| \le \nu} c_k e^{i\pi kt/\nu} \in B_\pi, \tag{4}$$

such that $\max_{\gamma \in \Gamma} |P(\gamma)| \le 1$ *and*

$$\max_{|t| \le \nu} |P(t)| \ge C \log \frac{2\nu}{\#\Gamma - 2\nu}. \tag{5}$$

2. We may assume that N is a large number. It is easy to see that it suffices to prove (3) for the case:

$$2N + N^{2/3} \le \#\Lambda \le \left(2 + \frac{1}{100}\right) N. \tag{6}$$

Using appropriate re-scaling, one can see that under condition (6), inequality (3) follows from the inequality:

$$K(\Lambda_N, B_{\pi/(1-\delta)}) \ge C \log \frac{2N}{\#\Lambda - 2N}, \tag{7}$$

where $0 < \delta < N^{-1/3}$.

3. To prove (7), we fix a number ν, $N - 2\sqrt{N} < \nu < N - \sqrt{N}$. Set

$$\Gamma := (\Lambda + 2\nu\mathbb{Z}) \cap [-\nu, \nu].$$

Without loss of generality, we may assume that $\#\Gamma = \#\Lambda$. Then, by Corollary 1, there is an exponential polynomial P satisfying (4), (5) and $|P(t)| \le 1$ on Γ, which implies $|P(t)| \le 1$ on Λ.

Denote by t_0 a maximum modulus point of P that lies on $[-\nu, \nu]$. We may assume that $P(t_0)$ satisfies:

$$|P(t_0)| = C \log \frac{2\nu}{\#\Gamma - 2\nu}, \tag{8}$$

where C is the constant in (5).

4. Set

$$h(t) := \frac{\sin \pi t}{\pi t}, \qquad g(t) := P(t)h\big(\nu^{-1/3}(t - t_0)\big).$$

Define δ by $1 + \nu^{-1/3} = 1/(1 - \delta)$. So, $\delta < N^{-1/3}$, as required. Then

$$g \in B_{\pi(1+\nu^{-1/3})} = B_{\pi/(1-\delta)}.$$

We see that $|g(t)| \le 1$ on Λ. The distance from t_0 to the points $\pm N$ is at least $\sqrt{N}$, so by (8), for all $t \ge N$ we get:

$$\big|g(t)\big| \le \big|P(t_0)\big|\big|h\big(\nu^{-1/3}(t - t_0)\big)\big| \le 1.$$

This gives (7). Proposition 1 is proved.

It is not difficult to deduce Theorem 1 from Proposition 1.

Theorem 2 is also an easy consequence of Proposition 1. Indeed, fix any function $\omega(\sigma) \uparrow \infty$ $(\sigma \uparrow \pi)$ and any sequence $\sigma_j > 0$ $(\sigma_j \uparrow \pi)$. Then it suffices to find a u.d. set Λ, $D^-(\Lambda) = 1$, such that $K(\Lambda, B_{\sigma_j}) > \omega(\sigma_{j+1})$, $j \in \mathbb{N}$. One may obtain such a set Λ as an infinite union of finite arithmetic progressions with differences π/σ_j, $j \in \mathbb{N}$. By Proposition 1, Λ will satisfy the property above provided these progressions are sufficiently long.

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