Functional analysis/Mathematical physics

The essential spectrum of $N$-body systems with asymptotically homogeneous order-zero interactions

Le spectre essentiel des systèmes à $N$-corps avec interactions asymptotiquement homogènes d’ordre zéro

Vladimir Georgescu $^a$, Victor Nistor $^{b,c}$

$^a$ Département de mathématiques, Université de Cergy-Pontoise, 95000 Cergy-Pontoise, France
$^b$ Université de Lorraine, UFR MIM, Ile du Saulcy, CS 50128, 57045 Metz cedex 01, France
$^c$ Pennsylvania State University, Mathematics Department, University Park, PA 16802, USA

A R T I C L E   I N F O

Article history:
Received 5 July 2014
Accepted after revision 15 September 2014
Available online 23 October 2014
Presented by the Editorial Board

A B S T R A C T

We study the essential spectrum of $N$-body Hamiltonians with potentials defined by functions that have radial limits at infinity. The results extend the HVZ theorem which describes the essential spectrum of usual $N$-body Hamiltonians. The proof is based on a careful study of algebras generated by potentials and their cross-products. We also describe the topology on the spectrum of these algebras, thus extending to our setting a result of A. Mageira. Our techniques apply to more general classes of potentials associated with translation invariant algebras of bounded uniformly continuous functions on a finite-dimensional vector space $X$.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Nous étudions le spectre essentiel des hamiltoniens des systèmes à $N$ corps avec potentiels définis par des fonctions qui ont des limites radiales à l’infini. Les résultats étendent le théorème HVZ, qui décrit le spectre essentiel des hamiltoniens des systèmes à $N$ corps usuels. La preuve de notre théorème principal est basée sur une étude approfondie des algèbres générées par les potentiels avec des limites radiales à l’infini et de leurs produits croisés. Nous décrivons également la topologie sur le spectre de ces algèbres, étendant ainsi à notre cas un résultat de A. Mageira. Nos techniques s’appliquent à des classes plus générales de potentiels associées à des algèbres de fonctions uniformément contiues bornées invariantes par translation.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

E-mail addresses: vladimir.georgescu@math.cnrs.fr (V. Georgescu), victor.nistor@univ-lorraine.fr (V. Nistor).

http://dx.doi.org/10.1016/j.crma.2014.09.029
1631-073X/© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
Version française abrégée

Soit $X$ un espace vectoriel réel de dimension finie et $S_X := (X \setminus \{0\}) / \mathbb{R}_+$ la sphère à l’infini de $X$. On dit qu’une fonction $v : X \to \mathbb{C}$ a des limites radiales uniformes à l’infini si $\lim_{\alpha \to \infty} v(\alpha) = 0$ existe uniformément en $\alpha \in S_X$. Soit $V_Y : X/Y \to \mathbb{R}$ une fonction borélienne ayant des limites radiales uniformes à l’infini, pour chaque sous-espace linéaire $Y \subset X$. Nous supposons $V_Y = 0$, sauf pour un nombre fini d’espaces $Y$. On note $\pi_Y$ la surjection canonique $X \to X/Y$ et on garde la notation $V_Y$ pour la fonction $V_Y \circ \pi_Y$. Dans cet article, nous utilisons des produits croisés de $C^*$-algèbres pour étudier le spectre essentiel des opérateurs de la forme $H := h(P) + \sum_{Y \supset \alpha} V_Y$, ici, $h : X^+ \to [0, \infty[$ est une fonction continue et propre et $P$ est l’observable moment $h(x) = -i \nabla X$. Soit $v : X \to \mathbb{C}$ et $a \neq 0$ tel que $\lim_{r \to \infty} vr(a + x)$ existe pour tout $x \in X$. Cette limite est une fonction $v \in X$, qui ne dépend que de la classe $\alpha := \alpha \circ \overline{a}$ dans $S_X$, que nous noterons $\tau_\alpha(v)$. Par exemple, si $v = V_Y$ avec $Y$ comme plus haut, alors $\tau_\alpha(V_Y) = V_Y$ si $\alpha \subset Y$ et $\tau_\alpha(V_Y) = V_Y(\pi_Y(\alpha)) \in \mathbb{R}$ si $\alpha \not\subset Y$, où $\pi_Y(\alpha) \in S_{X/Y}$ est naturellement défini. Plus tard (voir le Théorème 3.1), nous définirons $\tau_\alpha(S)$ pour une classe générale d’opérateurs $S$, en particulier pour $S = H$, ce qui donnera une nouvelle signification à la définition de $\tau_\alpha$.

Nous énonçons maintenant un cas particulier de notre résultat principal : si les fonctions $V_Y : X/Y \to \mathbb{R}$ sont bornées et ont des limites radiales uniformes à l’infini et si, pour chaque $\alpha \in S_X$, on pose $\tau_\alpha(H) = h(P) + \sum_{Y \supset \alpha} V_Y + \sum_{Y \supset \alpha} V_Y(\pi_Y(\alpha))$, alors le spectre essentiel de $H$ est $\sigma_{ess}(H) = \bigcup_{\alpha \in S_X} \sigma(\tau_\alpha(H))$.

1. Introduction

Let $X$ be a finite dimensional real vector space and, for each linear subspace $Y$ of $X$, let $V_Y : X/Y \to \mathbb{R}$ be a Borel function. We assume $V_Y = 0$, except for a finite number of $Y$. We keep the notation $V_Y$ for the function on $X$ given by $V_Y \circ \pi_Y$, where $\pi_Y : X \to X/Y$ is the natural map. In this paper, we use crossed-products of $C^*$-algebras to study the essential spectrum of Hamiltonians of the form $H := h(P) + \sum_{Y \subset \alpha} V_Y$, under certain conditions on the potentials $V_Y$. Here $h : X^+ \to [0, \infty[$ is a continuous, proper function and $P$ is the momentum observable (recall that proper means that $\lim_{k \to \infty} h(k) = +\infty$). More precisely, $h(P) = F^{-1} M_{\mathbb{R}} F$, where $F : L^2(X) \to L^2(X^*)$ is the Fourier transform and $M_{\mathbb{R}}$ is the operator of multiplication by $h$ (formally $P = -i \nabla$). Operators of this form cover the Hamiltonians that are currently the most interesting (from a physical point of view) Hamiltonians of $N$-body systems. There are two main examples. In a generalized version of the non-relativistic case, a scalar product is given on $X$, so, by taking $h(\xi) = \|\xi\|^2$, we get $h(P) = \Delta$, the positive Laplacian. In the simplest relativistic case, $X = (\mathbb{R}^3)^N$ and, writing the momentum $P = (P_1, \ldots, P_N)$, we have $h(P) = \sum_{k=1}^N (p_k^2 + m_k^2)^{1/2}$ for some real numbers $m_k$. We refer to [3] for a thorough introduction to the subject and study of these systems.

Let $S_X := (X \setminus \{0\}) / \mathbb{R}_+$ be the sphere at infinity of $X$, i.e. the set of all half-lines $\hat{a} := \mathbb{R}_+ a$. A function $v : X \to \mathbb{C}$ is said to have uniform radial limits at infinity if $v(\alpha) := \lim_{r \to \infty} v(ra + x)$ exists uniformly in $\alpha \in S_X$. From the definition of the topology on $S_X$, we get $v(\hat{a}) = \lim_{r \to \infty} v(ra + x)$, $\forall x \in X$. More generally, we are interested in functions $v$ such that $\lim_{r \to \infty} v(ra + x)$ exists for all $x \in X$. The limit may depend on $x$ and defines a function $\tau_\alpha(v) : X \to \mathbb{C}$, where $\alpha := \hat{a}$. For example, let us consider $v = V_Y$. Then $\tau_\alpha(V_Y)(x) = \lim_{r \to \infty} V_Y(r\pi_Y(\alpha) + x)$, $\forall x \in X$. In particular, $\tau_\alpha(V_Y) = V_Y$ whenever $\alpha := \hat{a} \subset Y$ (i.e. $a \in Y$). On the other hand, if $V_Y : X/Y \to \mathbb{C}$ has uniform radial limits at infinity and $\alpha \not\subset Y$, then $\tau_\alpha(V_Y) := \tau_\alpha(V_Y(\alpha)) \in S_{X/Y}$ is well defined and $\tau_\alpha(V_Y)(x) = \tau_\alpha(V_Y(\pi_Y\alpha))(x)$ turns out to be a constant.

Theorem 1.1. Let $V_Y : X/Y \to \mathbb{R}$ be bounded with uniform radial limits at infinity. If $\alpha \in S_X$ set

$$
\tau_\alpha(H) = h(P) + \sum_{Y \supset \alpha} V(Y) = h(P) + \sum_{Y \supset \alpha} V_Y + \sum_{Y \supset \alpha} V_Y(\pi_Y(\alpha)).
$$

Then $\sigma(\tau_\alpha(H)) = \{c_\alpha, \infty\}$ for some real $c_\alpha$ and $\sigma_{ess}(H) = \bigcup_{\alpha \in S_X} \sigma(\tau_\alpha(H)) = \inf c_\alpha c_\alpha, \infty$.}

Here $\bigcup_{\alpha}$ is the closure of the union. Sometimes the union is already closed [11]. Unbounded potentials are considered in Theorem 3.2. If all the radial limits are zero, which is the case of the usual $N$-body potentials, then the terms corresponding to $\alpha \not\subset Y$ are dropped in Eq. (1). Consequently, if $h(P) = \Delta$ is the non-relativistic kinetic energy, we recover the Hunziker, van Winter, Zhishlin (HVZ) theorem. Descriptions of the essential spectrum of various classes of self-adjoint operators in terms of limits at infinity of translates of the operators have already been obtained before, see for example [7,12,8] (in historical order). Our approach is based on the “localization at infinity” technique developed in [5,6] in the context of crossed-product $C^*$-algebras.

Let us sketch the main idea of this approach. Let $C_0^b(X)$ be the algebra of bounded uniformly continuous functions, $C_0(X)$ the ideal of functions vanishing at infinity, and $C(X^+) = C + C_0(X)$. Consider a translation invariant $C^*$-subalgebra $\mathcal{A} \subset C_0^b(X)$ containing $C(X^+)$ and let $A$ be its character space. Note that $A$ is a compact topological space that naturally contains $X$ as an open dense subset and $\delta(A) = A \setminus X$ can be thought of as a boundary of $X$ at infinity. Recall that a self-adjoint operator $H$ on a Hilbert space $\mathcal{H}$ is said to be affiliated to a $C^*$-algebra $\mathcal{A}$ of operators on $\mathcal{H}$ if one has $(H + 1)^{-1} \in \mathcal{A}$. Then with each self-adjoint operator $H$ affiliated to the crossed product $\mathcal{A} \times X$ by the action of $X$, one may associate a family of self-adjoint operators $H_x$ affiliated to $\mathcal{A} \times X$ indexed by the characters $x \in \delta(A)$. This family
completely describes the image of \( H \) (in the sense of affiliated operators) in the quotient of \( \mathcal{A} \times X \) with respect to the ideal of compact operators. In particular, the essential spectrum of \( H \) is the closure of the union of the spectra of the operators \( H_x \). These operators are the \textit{localizations at infinity} of \( H \), more precisely, \( H_x \) is the \textit{localization of \( H \) at point} \( x \).

Once chosen the algebra \( \mathcal{A} \), in order to use these techniques of this paper, we also need: (1) to have a good description of the character space of the Abelian algebra \( \mathcal{A} \), and (2) to have an efficient criterion for affiliation to the crossed product \( \mathcal{A} \times X \). We also indicate how to achieve (1) and (2).

2. Crossed products and localizations at infinity

For \( p \in X^* \) and \( q \in X \) let \((S_p,f)(x) = e^{i(xp)}f(x)\) and \((T_qf)(q) = f(x + q)\). We say that \( A \in B(L^2(X)) \) has \textit{the position-momentum limit property} if \( \lim_{q \to 0} \| [S_p, A] \| = 0 \) and \( \lim_{q \to 0} \| (T_q - 1)A \| = 0 \) (where \( A \) is the relation holds for \( A \) and \( A^* \)). The set of such operators is a \( C^* \)-algebra equal to the crossed product \( C_0(X) \rtimes X \) [5]. Note that if \( \mathcal{A} \) is a translation invariant \( C^* \)-subalgebra of \( C_0(X) \), then there is a natural realization of the abstract crossed product \( \mathcal{A} \times X \) as a \( C^* \)-algebra of operators on \( L^2(X) \) and we do not distinguish the two algebras. We describe this concrete version of \( \mathcal{A} \times X \) below.

If \( \varphi : X \to \mathbb{C} \) and \( \psi : X^* \to \mathbb{C} \) are measurable functions, then \( \varphi(q) \) and \( \psi(P) \) are the operators on \( L^2(X) \) defined as follows: \( \varphi(q) \) acts as multiplication by \( \varphi \) and \( \psi(P) = F^{-1}M_{\varphi}F \), where \( F \) is the Fourier transform \( L^2(X) \to L^2(X^*) \) and \( M_{\varphi} \) is the operator of multiplication by \( \varphi \). Then \( \varphi \mapsto \psi(P) \) is an isomorphism between \( C_0(X) \) and the \( C^* \)-algebra \( C^*(X^*) \) and \( \mathcal{A} \times X \) is the norm closed linear space of bounded operators on \( L^2(X) \) generated by the products \( \varphi(q) \psi(P) \) with \( \varphi \in \mathcal{A} \) and \( \psi \in C_0(X^*) \). In particular, \( \mathcal{A} \times X \) consists of operators that have the position-momentum limit property.

We recall the definition of localizations at infinity for such operators. Assume \( \mathcal{C}(X^+) \subset \mathcal{A} \), so \( \mathcal{A} \) is a compactification of \( X \) and \( \delta(A) = \{ x \mid X, |x| \} \) is a compact. If \( q \in X \) and \( \varphi \) is a function on \( X \) then \( T_q \varphi \) is its translation by \( q \). We extend this definition of \( T_q \) by replacing \( q \in X \) with \( x \in \mathcal{A} \): \( T_q \varphi(x) = \varphi(T_x \varphi) \), for any \( \varphi \in \mathcal{A} \), \( x \in \mathcal{A} \), and, \( x \in X \). It is clear that \( T_q \varphi \in C_0(X) \) and that its definition coincides with the previous one if \( x = q \in X \). Moreover, we also get “translations at infinity” of \( \varphi \in \mathcal{A} \) by elements \( x \in \delta(A) \); note however that such a translation does not belong to \( \mathcal{A} \) in general. Also, the function \( \varphi \mapsto T_q \varphi \) defined on \( \mathcal{A} \) is continuous if \( C_0(X) \) is equipped with the topology of local uniform convergence, hence \( T_q \varphi(q) = \lim_{q \to x} T_q \varphi \) in this topology for any \( x \in \delta(A) \). If \( A \) is an operator on \( L^2(X) \), let \( T_q(A) = T_q HA \) be its translation by \( q \in X \). Clearly \( T_q \varphi \) is a \( \mathcal{A} \) operator and, when \( \mathcal{A} \) is \( \mathcal{A} \times X \), we may also consider “translations at infinity” by elements of the boundary \( \delta(A) \) of \( X \) in \( \mathcal{A} \) and we get a useful characterization of the compact operators. The following are mainly consequences of \( [6, \text{Theorem 11.5}] \):

(i) For each \( x \in \mathcal{A} \), there is a unique morphism \( \tau_x : \mathcal{A} \rtimes X \to C_0(X^+) \times X \) such that \( \tau_x(\varphi(q) \psi(P)) = (T_x \varphi(q)) \psi(P) \), \( \psi \in C_0(X^+) \), \( \varphi \in \mathcal{C}(X^+) \). (ii) \( \bigcap_{x \in \delta(A)} \ker \tau_x = C_0(X^+) \times X \equiv X^* \times X \) is ideal of compact operators on \( L^2(X) \). (iii) If \( H \) is a self-adjoint operator on \( L^2(X) \) affiliated to \( \mathcal{A} \) then for each \( x \in \delta(A) \) the limit \( \tau_x(H) = s-lim_{q \to x} T_q HT_q^* \) exists and \( \sigma_{ess}(H) = \bigcup_{x \in \delta(A)} \sigma(\tau_x(H)) \).

To be precise, the last strong limit means: \( \tau_x(H) \) is a self-adjoint operator (not necessarily densely defined) on \( L^2(X) \) and \( s-lim_{q \to x} \theta(T_qHT_q^*) = \theta(\tau_x(H)) \) for all \( \theta \in C_0(\mathbb{R}) \). It is clear in the last three statements above one may replace \( \delta(A) \) by a subset \( \pi \) if for each \( A \in \mathcal{A} \) we have: \( \tau_x(A) = 0 \Rightarrow \forall x \in \pi \Rightarrow \tau_x(A) = 0 \forall x \in \delta(A) \). In the case of groupoid (pseudo)algebras (that is, when \( \mathcal{A} \) is a manifold with corners), the morphisms \( \tau_x \) can be defined using restrictions to fibers, as in \([9]\), and the last three statements above (i)–(iii) remain valid.

3. Main results

As a warm-up and in order to introduce some general notation, we treat first the two-body case, where complete results may be obtained by direct arguments. The algebra of interactions in the standard two-body case is \( \mathcal{C}(X^+) \), and hence the Hamiltonian algebra is

\[
\mathcal{C}(X^+) \times X = C \times X + C_0(X) \times X = C^*(X^+) \times X^*
\]

where the sums are direct. Thus \( \mathcal{C}(X^+) \times X \equiv X^* \times X \) is \( C^*(X) \), which finishes the theory. Another elementary case, which has been considered as an example in [5], is \( X = \mathbb{R} \) with \( \mathcal{C}(\mathbb{R}) \) replaced by the algebra \( \mathcal{C}(\mathbb{R}) \) of continuous functions that have limits (distinct in general) at \( \pm \infty \). Then there is no natural direct sum decomposition of \( \mathcal{C}(\mathbb{R}) \times \mathbb{R} \) as in (2), but one has, by standard arguments, \( \mathcal{C}(\mathbb{R}) \times \mathbb{R} \equiv \mathcal{C}(\mathbb{R}) \times \mathcal{C}(\mathbb{R}) \). Our purpose in this section is to extend this equation to arbitrary \( X \).

Let \( \mathcal{C}(X) \) be the closure in \( C_0(X) \) of the subalgebra of functions homogeneous of degree zero outside a compact set. Then \( \mathcal{C}(X) = \{ u \in C_0(X) \mid \lim_{x \to X} u(x) \} \) exists uniformly in \( \hat{a} \in \mathbb{S}X \), where, we recall, \( \hat{a} := \mathbb{R}_+ \hat{a} \) and \( \mathbb{S}X := (X \setminus \{0\})/\mathbb{R}_+ \), so \( \hat{a} \in \mathbb{S}X \). As a set, the character space of \( \mathcal{C}(X) \) can be identified with the disjoint union \( X = X \cup \mathbb{S}X \). The topology induced by the character space on \( X \) is the usual one and the intersections with \( X \) of the neighborhoods of some \( a \in \mathbb{S}X \) are the sets that contain a truncated cone \( C \) such that there is a \( a \in \alpha \) such \( \alpha \in \alpha \) if \( \lambda \geq 1 \). The set of such subsets is a filter \( \mathcal{A} \) on \( X \) and, if \( Y \) is a Hausdorff space and \( u : X \to Y \), then \( \lim_{u \to X} u = y \) means that \( u^{-1}(V) \in \mathcal{A} \) for any neighborhood \( V \) of \( u \). We shall
write \( \lim_{t \to 0} u(x) \) instead of \( \lim u \). We have that \( C(\overline{X}) \) is a translation invariant \( C^* \)-subalgebra of \( C^b(\overline{X}) \) and so the crossed product \( C(\overline{X}) \rtimes X \) is well defined. We have the following explicit description of this algebra.

**Proposition 3.1.** The algebra \( C(\overline{X}) \rtimes X \) acting on \( L^2(X) \) consists of bounded operators \( A \) that have the position-momentum limit property and are such that the limit \( \tau_A(A) = \lim_{t \to 0} T_a A T_a^* \) exists for each \( \alpha \in S_X \). If \( A \in C(\overline{X}) \rtimes X \) and \( \alpha \in S_X \), then \( \tau_A(A) \in C^*(\overline{X}) \) and \( \tau : A \mapsto \tau_A(A) \) is norm continuous. The map \( \tau : C(\overline{X}) \rtimes X \to C(S_X) \otimes C^*(\overline{X}) \) is a surjective morphism whose kernel is the set of compact operators on \( L^2(X) \), which gives \( C(\overline{X}) \times X/\mathcal{N}(X) \cong C(S_X) \otimes C^*(\overline{X}) \). If \( H \) is a self-adjoint operator affiliated to \( C(\overline{X}) \rtimes X \) then \( \tau_A(H) = \lim_{t \to 0} T_a H T_a^* \) exists for all \( \alpha \in S_X \) and \( \sigma_{ess}(H) = \bigcup \alpha \sigma(\tau_A(H)) \).

In the next two examples \( H = h(P) + V \) with \( h : X^* \to [0, \infty) \) continuous and proper. We denote by \( | \cdot | \) a fixed norm on \( X^* \) and by \( H \) we denote the usual Sobolev spaces on \( X \) (\( s \in \mathbb{R} \)).

**Example 1.** Let \( V \) be a bounded symmetric operator satisfying: (1) \( \lim_{p \to 0} ||[S_p, V]|| = 0 \) and (2) the limit \( \tau_A(V) = s \lim_{t \to 0} T_a V T_a^* \) exists for each \( \alpha \in S_X \). Then \( H \) is affiliated to \( C(\overline{X}) \rtimes X \) and \( \tau_A(H) = h(P) + \tau_A(V) \). Moreover, if \( V \) is a function, then \( \tau_A(V) \) is a number, but in general we have \( \tau_A(V) = v_A(P) \) for some function \( v_A \in C^b(\mathbb{R} \times X) \).

**Example 2.** Assume that \( h \) is locally Lipschitz and that there exist \( c, s > 0 \) such that, for all \( p \) with \( |p| > 1, |\nabla h(p)| \leq c(1 + h(p)) \) and \( c^{-1} |p|^s \leq (1 + h(p))^{1/2} \leq |p|^s \). Let \( V : H^s \to H^{-s} \) such that \( \pm V \leq \mu h(p) + v \) for some numbers \( \mu, v \) with \( \mu < 1 \) and satisfying the next two conditions: (1) \( \lim_{p \to 0} ||[S_p, V]||_{H^{-s} \to H^{-s}} = 0 \), (2) \( \forall \alpha \in S_X \) the limit \( \tau_A(V) = s \lim_{t \to 0} T_a V T_a^* \) exists strongly in \( B(H^s, H^{-s}) \). Then \( h(P) + V \) and \( h(P) + \tau_A(V) \) are symmetric operators \( H^s \to H^{-s} \) that induce self-adjoint operators \( H \) and \( \tau_A(H) \) in \( L^2(X) \) affiliated to \( C(\overline{X}) \rtimes X \) and \( \sigma_{ess}(H) = \bigcup \alpha \sigma(\tau_A(H)) \).

We now treat the \( N \)-body case. We first indicate a general way of constructing \( N \)-body Hamiltonians. For each linear subspace \( Y \subset X \), let \( A(X, Y) \subset C^b(\mathbb{R} \times X) \) be a translation invariant \( C^* \)-subalgebra containing \( C_0(X, Y) \) with \( A(X) = A(0) = C \). We embed \( A(X, Y) \subset C^b(\mathbb{R} \times Y) \) as usual by identifying \( v \) with \( v \circ \tau_Y \). Then the \( C^* \)-algebra \( A \) generated by these algebras is a translation invariant \( C^* \)-subalgebra of \( C^b(\mathbb{R} \times Y) \) containing \( C(X^*) \) and thus we may consider the crossed product \( A \rtimes X \) which is equal to the \( C^* \)-algebra generated by the crossed products \( A(X, Y) \rtimes Y \). The operators affiliated to \( A \rtimes X \) are \( N \)-body Hamiltonians. The standard \( N \)-body algebra corresponds to the minimal choice \( A(X, Y) = C_0(X, Y) \) and has remarkable properties, which makes its study relatively easy (it is graded by the lattice of subspaces of \( X \)). Our purpose in this paper is to study what could arguably be considered to be the simplest extension of the classical \( N \)-body obtained by choosing \( A(X, Y) = C(\overline{X} \setminus Y) \) for all \( Y \). The next more general case would correspond to the choice \( A(X, Y) = \mathcal{V}(X \setminus Y) \) (slowly oscillating functions, i.e., the closure in sup norm of the set of bounded functions of class \( C^1 \) with derivatives tending to zero at infinity).

**Definition 3.2.** Let \( E(X) \) be the \( C^* \)-subalgebra of \( C^b(\mathbb{R} \times X) \) generated by \( \bigcup_{Y \subset X} C(\overline{X} \setminus Y) \).

Clearly \( E(X) \) is a translation invariant \( C^* \)-subalgebra of \( C^b(\mathbb{R} \times X) \) containing \( C(X^*) := C_0(X) + C \). If \( Y \) is a linear subspace of \( X \) then \( E(X) \subset C^b(X, Y) \) is well defined and naturally embedded in \( E(X) \): it is the \( C^* \)-algebra generated by \( \bigcup_{Y \subset X} C(\overline{X} \setminus Y) \). We have \( C = C(0) = E(0) \subset C(X, Y) \subset C(\mathbb{R} \times X) \subset C^b(X, Y) \). If \( \alpha \in S_X \), we denote by abuse of notation \( X/\alpha \) be the quotient \( X/\alpha \) by the subspace \( |\alpha| := \mathbb{R} \alpha \) generated by \( \alpha \) and let us set \( \pi_\alpha = \pi_{|\alpha|} \). It is clear that \( \tau_\alpha(u)(x) = \lim_{\alpha \to X} u(\alpha + x) \) exists \( \forall u \in E(X) \) and that the resulting function \( \tau_\alpha(u) \) belongs to \( E(X) \). The map \( \tau_\alpha \) is an endomorphism of \( E(X) \) and a linear projection of \( E(X) \) onto the \( C^* \)-subalgebra \( E(X)/\alpha \).

If \( \alpha \in S_X \) and \( \beta \in S_{X/\alpha} \), then \( \beta \) generates a one-dimensional linear subspace \( \beta \) := \( \mathbb{R} \beta \subset X/\alpha \), as above, and hence \( \pi^{-1}(\beta \beta) \) is a two-dimensional subspace of \( X \) that we shall denote \( \beta \). We shall identify \( X/\alpha \) with \( X/\alpha \). Then we have two idempotent morphisms \( \tau_\beta : E(X) \to E(X/\alpha) \) and \( \tau_\beta : E(X/\alpha) \to E(X/\alpha/\beta) \). Thus \( \tau_\beta \tau_\alpha : E(X) \to E(X/\alpha/\beta) \) is an idempotent morphism. This construction extends in an obvious way to families \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( n \leq \dim X \) and \( \alpha_1 \in S_X, \alpha_2 \in S_{X/\alpha_1}, \alpha_3 \in S_{X/\alpha_1/\alpha_2}, \ldots \) (we allow \( n = 0 \) and denote \( A \) the set of all such families). The endomorphism \( \tau_\alpha \) of \( E(X) \) is defined by induction: \( \tau_\alpha = \tau_\beta \tau_\alpha \). We also define \( \tau_\alpha(\alpha_1, \ldots, \alpha_n) \) by induction, so this is an \( n \)-dimensional subspace of \( X \) associated with \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and we denote \( X/\alpha \) the quotient of \( X \) with respect to it. Thus \( \tau_\alpha \) is an endomorphism of \( E(X) \) and a projection of \( E(X) \) onto \( E(X/\alpha) \).

**Proposition 3.3.** If \( \alpha \in A \) and \( a \in X/\alpha \), then \( x(u) = (\tau_\alpha u)(a) \) defines a character of \( E(X) \). Conversely, each character of \( E(X) \) is of this form.

**Remark 1.** A natural Abelian \( C^* \)-algebra in the present context is the set \( \mathcal{R}(X) \) of all bounded uniformly continuous functions \( v : X \to C \) such that \( \lim_{x \to a} v(x + a) \) exists locally uniformly in \( x \in X \) for each \( a \in X \). It would be interesting to find an explicit description of its spectrum.

This description of the spectrum of \( E(X) \) extends [10]. We now state our main results.
Theorem 3.1. Let $H$ be a self-adjoint operator on $L^2(X)$ affiliated to $E(X) \times X$. Then for any $a \in X \setminus \{0\}$ the limit $s$-$\lim_{r \to +\infty} T_{ra} H T_{ra}^* =: \tau_a(H)$ exists and $\sigma_{\text{ess}}(H) = \bigcup_{a \in S_X} \sigma(\tau_a(H))$.

Theorem 3.2. Let $h$ be as in Example 2 and $V = \sum V_y$ with $V_Y : \mathcal{H}^\ell \to \mathcal{H}^{\ell-1}$ symmetric operators such that $V_Y = 0$ but for a finite number of $Y$ and satisfying: (i) $\exists \mu_y, v_Y \geq 0$ with $\sum \mu_Y < 1$ such that $\pm V_Y \leq \mu_Y h(P) + v_Y$, (ii) $\lim_{p \to 0} \left| \left| S_P, V_Y \right| \right|_{\mathcal{H}^{\ell-1}} = 0$, (iii) $|T_{\alpha}, V_Y| = 0$ for all $y \in Y$, (iv) $\tau_a(V_Y) := s$-$\lim_{\alpha \to a} \tau_a V_Y T_{\alpha} T_{\alpha}^*$ exists in $B(\mathcal{H}^{\ell}, \mathcal{H}^{\ell-1})$ for all $\alpha \in S_{X/Y}$. Then the maps $\mathcal{H}^{\ell} \to \mathcal{H}^{\ell-1}$ given by $h(P) + V$ and $h(P) + \sum Y \tau_a(V_Y)$ induce self-adjoint operators $H$ and $\tau_a(H)$ in $L^2(X)$ affiliated to $E(X)$ and $\sigma_{\text{ess}}(H) = \bigcup_{\alpha \in S_X} \sigma(\tau_a(H))$.

Example 3. Using [2], we also obtain that Theorem 3.2 covers uniformly elliptic operators of the form $H = \sum_{|\ell| \leq s} p^k a_{\ell k} P^\ell$, where $a_{\ell k}$ are finite sums of functions of the form $v_Y : X/Y \to \mathbb{R}$ bounded measurable such that $\lim_{y \to 0} v_Y(2)$ exists uniformly in $\alpha \in S_{X/Y}$. The fact that we allow $a_{\ell k}$ to be only bounded measurable for $|k| = |\ell| = s$ is not trivial.

In addition to the above-mentioned results, we also use general results on cross-product $C^*$-algebras, their ideals, and their representations [4,13]. The maximal ideal spectrum of the algebra $E(X)$ is of independent interest and can be used to study the regularity properties of the eigenvalues of the $N$-body Hamiltonian [1]. Its relation to the constructions of Vasy in [14] will be studied elsewhere.

Acknowledgements

We thank Bernd Ammann for several useful discussions. Victor Nistor was partially supported by ANR SINGSTAR 2014–18 and by NSF Grant DMS-1016556.

References