Algebraic geometry

Geometric construction of generators of CoHA of doubled quiver

Construction géométrique des générateurs de l’algèbre cohomologique de Hall du double d’un carquois

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ABSTRACT

Let $Q$ be the double of a quiver. According to Efimov, Kontsevich and Soibelman, the cohomological Hall algebra (CoHA) associated with $Q$ is a free super-commutative algebra. In this short note, we confirm a conjecture of Hausel, which gives a geometric realisation of the generators of the CoHA.

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RÉSUMÉ

Soit $Q$ le double d’un carquois. Selon Efimov, Kontsevich et Soibelman, l’algèbre cohomologique de Hall (CoHA) associée à $Q$ est une algèbre libre super-commutative. Dans cette note, nous démontrons la conjecture de Hausel, donnant une réalisation géométrique des générateurs de cette algèbre.

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1. Introduction

Let $Q = (I, Ω)$ be a quiver with a set of vertices $I$ and with $a_{ij}$ arrows from $i \in I$ to $j \in I$. For each dimension vector $γ = (γ^i)_{i \in I} \in \mathbb{Z}^I_{≥0}$, we have the affine $ℂ$-variety $M_γ$ of representations of $Q$ in complex coordinate space $\bigoplus_{i \in I} ℂ^{γ^i}$. It is acted on by the complex algebraic group $G_γ = \prod_{i \in I} \GL_{γ^i}(ℂ)$ and the action factors through $PG_γ := G_γ/Γ_m$, where $Γ_m$ is embedded diagonally in $G_γ$. We denote by $[M_γ/G_γ]$ the moduli stack of representations of $Q$ of dimension $γ$.

For $γ_1, γ_2 \in \mathbb{Z}^I_{≥0}$, let $M_{γ_1, γ_2}$ be the subvariety of $M_{γ_1+γ_2}$ consisting of the representations of $Q$ such that the subspace $\bigoplus_{i \in I} ℂ^{γ^i} \subset \bigoplus_{i \in I} (ℂ^{γ^i} \oplus ℂ^{γ^i})$ forms a sub-representation. Let $G_{γ_1, γ_2} \subset G_{γ_1+γ_2}$ be the subgroup preserving $\bigoplus_{i \in I} ℂ^{γ^i}$. Then $[M_{γ_1, γ_2}/G_{γ_1, γ_2}]$ is the moduli stack classifying the extensions of representations of $Q$ of dimension vector $γ_2$ by that of dimension vector $γ_1$. We have the correspondence:

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from which Kontsevich and Soibelman [8] have constructed an associative algebra structure on
\[ \mathcal{H} := \bigoplus_{\gamma \in \mathbb{Z}_{\leq 0}} \mathcal{H}_\gamma, \quad \mathcal{H}_\gamma := H^*([M_\gamma/G_\gamma]) = H^*_C(M_\gamma), \]
which is called the cohomological Hall algebra (CoHA) of the quiver \( Q \). Here and after, all the cohomological groups take coefficients in \( Q \). The resulting product has a shift in cohomological degree:
\[ H^*_C(M_\gamma) \times H^*_C(M_{\gamma'}) \rightarrow H^*_{C_{\gamma\gamma'}}(M_{\gamma+\gamma'}), \]
where
\[ \chi_Q(\gamma_1, \gamma_2) = \sum_{i \in I} \gamma_1^i \gamma_2^i - \sum_{i,j \in I} a_{i,j} \gamma_1^i \gamma_2^j. \]

Suppose that the quiver \( Q \) is symmetric, i.e. \( a_{ij} = a_{ji} \). In this case, \( \mathcal{H} \) has more structures. First of all, one can make \( \mathcal{H} \) into a \((\mathbb{Z}_{\geq 0}, \mathbb{Z})\)-graded algebra, by requiring elements in \( H^k_{C_{\gamma}}(M_\gamma) \) to be of bidegree \((\gamma, k + \chi_Q(\gamma, \gamma'))\). Secondly, following Efimov [5], we can twist the multiplication by a sign such that \((\mathcal{H}, \ast)\) is a super-commutative algebra with respect to the \( \mathbb{Z} \)-grading. In fact, for \( a_{\gamma, k} \in \mathcal{H}_{\gamma, k}, a_{\gamma', k} \in \mathcal{H}_{\gamma', k} \), we have:
\[ a_{\gamma, k} a_{\gamma', k'} = (-1)^{\chi_Q(\gamma, \gamma')} a_{\gamma', k} a_{\gamma, k'}. \]

We can find a bilinear form \( \psi : (\mathbb{Z}/2)^I \times (\mathbb{Z}/2)^I \rightarrow \mathbb{Z}/2 \) such that
\[ \psi(\gamma_1, \gamma_2) + \psi(\gamma_2, \gamma_1) = \chi_Q(\gamma_1, \gamma_2) + \chi_Q(\gamma_2, \gamma_1) \chi_Q(\gamma_1, \gamma_2) \quad \text{mod} \ 2. \]

Then the twisted product on \( \mathcal{H} \) is defined to be
\[ a_{\gamma, k} a_{\gamma', k'} = (-1)^{\chi_Q(\gamma, \gamma')} a_{\gamma', k} a_{\gamma, k'}, \]
where \( \bar{\gamma} \) is the image of \( \gamma \) in \((\mathbb{Z}/2)^I\).

For the symmetric quiver \( Q \), it is conjectured by Kontsevich and Soibelman [8] and proved by Efimov [5] that the \((\mathbb{Z}_{\geq 0}, \mathbb{Z})\)-graded algebra \((\mathcal{H}, \ast)\) is a free super-commutative algebra generated by a \((\mathbb{Z}_{\geq 0}, \mathbb{Z})\)-graded vector space \( V \) of the form \( V = V^{\text{prim}} \otimes \mathbb{Q}[x] \), with \( x \) an element of degree \((0,2)\), and for all \( \gamma \in \mathbb{Z}_{\geq 0} \) the vector space \( V^{\text{prim}}_{\gamma, k} \) is non-zero for only finitely many \( k \). Geometrically, via the isogeny \( G_\gamma \rightarrow P G_\gamma \times \mathbb{G}_m \), we have
\[ \mathcal{H}_\gamma = H^*_C(M_\gamma) \cong H^*_{C_{\gamma, k}}(M_\gamma) \otimes H^*_G(\text{pt}), \]
and it gives the above factorisation \( V = V^{\text{prim}} \otimes \mathbb{Q}[x] \). Let \( M^{\text{st}}_\gamma \) be the stable part of \( M_\gamma \) with respect to the 0-stability condition, i.e. \( M^{\text{st}}_\gamma \) is the free quotient of \( M^q_{\gamma} \) by all the deformations \( M^q_{\gamma} \). Then as we will see in Theorem 2.2, \( V^{\text{prim}}_{\gamma, k} \) is in fact the pure part of \( H^*(M^{\text{st}}_\gamma/\mathbb{P} G_\gamma) \), which we say “pure” refers to the mixed Hodge structure on the cohomological group. Let \( c_{\gamma, k} = \dim V^{\text{prim}}_{\gamma, k} \), the above result implies that the quantum Donaldson–Thomas invariants of the quiver \( Q \) without potential and with 0-stability condition is:
\[ \Omega(\gamma)(q) = \sum_{k \in \mathbb{Z}} c_{\gamma, k} q^{k/2} \in \mathbb{Z}[q^{1/2}]. \]

In particular, the coefficients are positive.

On the other hand, in the work of Hausel, Letellier and Rodriguez-Villegas [6], they found another expression for the quantum Donaldson–Thomas invariants. From now on, we work with quivers \( Q = (I, \Omega) \) that are the double of another quiver, i.e. \( \Omega = \Omega_0 \uplus \Omega^{op}_0 \), where \( \Omega^{op}_0 \) is obtained by reversing all the arrows in \( \Omega_0 \). In this case, \( M_\gamma \) is endowed with a \( G_\gamma \)-invariant holomorphic symplectic form \( \omega \). Let \( \mu : M_\gamma \rightarrow \mathbb{Q}_{0}^{\gamma} \) be the corresponding moment map, here \( \mathbb{Q}_{0}^{\gamma} \) is the trace 0 part of \( g_{\gamma} : = \text{Lie}(G_\gamma) \). Let \( C \) be the \( G_\gamma \)-orbit of a generic (as to be explained below) regular semisimple element in \( g_{\gamma}^{0} \). The group \( P G_\gamma \) acts freely on \( \mu^{-1}(C) \) and we have the geometric quotient \( \mu^{-1}(C)/P G_\gamma \), which is smooth quasi-projective algebraic variety. Furthermore, the Weyl group \( W_\gamma \) of \( G_\gamma \) acts on the cohomological groups \( H^*(\mu^{-1}(C)/P G_\gamma) \). One of the main results of [6] states that:
\[ \Omega(\gamma)(q) = q^{1/2} \chi_Q(\gamma) \sum_i \dim(H^2(\mu^{-1}(C)/P G_\gamma)^{W_\gamma}) q^i. \]
Based on this result, Hausel conjectured that the cohomological groups
\[ H^k\left(\mu^{-1}(\mathcal{O})/PG_Y\right)^W, \]
are geometric realisations of the generating set \( V_{\gamma,k+\chi_0(\gamma,y)}. \) (Of course, the conjecture is meaningful only when \( \Omega(\gamma)(q) \) is non-zero. We will always impose this condition.) In this article, we confirm this conjecture. Our construction goes as follows:

Let \( \chi : \gamma^0 \to \gamma^0, \gamma^0 \equiv \gamma^0 \equiv W_Y \) be the characteristic morphism. Following Ginzburg [7], we consider the composition \( f : M_Y \to \gamma^0 \to \gamma^0 \equiv W_Y. \) Let
\[ t^0_{\gamma} \geq \sum_{\gamma_1, \gamma_2 \in \mathbb{Z} \geq 0} W_Y \cdot (t^0_{\gamma_1} \otimes t^0_{\gamma_2}), \]
we call conjugates of elements in it generic regular semisimple elements. Let
\[ U_Y := f^{-1}(t^0_{\gamma} / W_Y), \]
then the group \( PG_Y \) acts freely on \( U_Y \) and the quotient \( U_Y / PG_Y \) is a quasi-projective algebraic variety. Furthermore, the restriction of the morphism \( f \) to \( U_Y \) descends to a morphism \( \tilde{f} : U_Y / PG_Y \to t^0_{\gamma} / W_Y. \) We will prove that it makes \( U_Y / PG_Y \) a fiber bundle on \( t^0_{\gamma} / W_Y \) with fibers isomorphic to \( \mu^{-1}(\mathcal{O})/PG_Y. \) Our main result is the following:

**Theorem 1.1.** The pure part of \( H^*(U_Y / PG_Y) \) is equal to \( H^*(\mu^{-1}(\mathcal{O})/PG_Y)^W. \) The restriction \( H^*_{PG_Y}(M_Y) \to H^*(U_Y / PG_Y) \) factors through and is surjective onto the pure part of the latter, and its restriction to \( V^0_{\gamma,k+\chi_0(\gamma,y)} \) induces an isomorphism
\[ V^0_{\gamma,k+\chi_0(\gamma,y)} \cong H^*(\mu^{-1}(\mathcal{O})/PG_Y)^W. \]

2. Proof of the main theorem

We begin by recalling the construction of Kontsevich and Soibelman of the cohomological Hall algebra. Given two vectors \( \gamma_1, \gamma_2 \in \mathbb{Z} \geq 0, \) let \( Y = \gamma_1 + \gamma_2. \) The product \( H_{\gamma_1} \times H_{\gamma_2} \to H_Y \) is defined to be the composition of the Künneth isomorphism:
\[ H^*_{G_{\gamma_1}}(M_{\gamma_1}) \otimes H^*_{G_{\gamma_2}}(M_{\gamma_2}) \cong H^*_{G_{\gamma_1} \times G_{\gamma_2}}(M_{\gamma_1} \times M_{\gamma_2}), \]
and of the following morphisms:
\[ H^*_k(\gamma_1, \gamma_2)_2 (M_{\gamma_1} \times M_{\gamma_2}) \cong H^*_k(\gamma_1, \gamma_2)_2 (M_{\gamma_1}, \gamma_2) \phi_1 \to H^*_k(\gamma_1, \gamma_2)_2 (M_{\gamma_1}, \gamma_2) \phi_2 \to H^*_{Y_2}(M_Y), \]
where \( c_1 = \dim_c M_{\gamma_1} - \dim_c M_{\gamma_1, \gamma_2} \) and \( c_2 = \dim_c G_{\gamma_1, \gamma_2} - \dim_c G_{\gamma_2}, \) and the first isomorphism is induced by the fibrations in affine spaces:
\[ M_{\gamma_1, \gamma_2} \to M_{\gamma_1} \times M_{\gamma_2}, \quad G_{\gamma_1, \gamma_2} \to G_{\gamma_1} \times G_{\gamma_2}, \]
and the other morphisms \( \phi_1, \phi_2 \) are natural push forwards.

**Lemma 2.1.** Under the restriction \( H^*([M_Y / G_Y]) \to H^*([M_Y^0 / G_Y]), \) the image of
\[ \bigoplus_{\gamma_1, \gamma_2 \in \mathbb{Z} \geq 0} H_{\gamma_1} \times H_{\gamma_2} \]
in \( H_Y = H^*([M_Y / G_Y]) \) goes to 0.

**Proof.** By the definition of Gysin map, the morphism \( \phi_1, \) in composition \( (1) \) factorises as
\[ H^*([M_{\gamma_1, \gamma_2} / G_{\gamma_1, \gamma_2}]) \to H^*([M_{\gamma_1} / G_{\gamma_1, \gamma_2}] + 2c_1) \to H^*([M_{\gamma_1, \gamma_2} / G_{\gamma_1, \gamma_2}]). \]
Using the long exact sequence
\[ \cdots \to H^*([M_{\gamma_1, \gamma_2} / G_{\gamma_1, \gamma_2}]) \to H^*([M_{\gamma_1} / G_{\gamma_1, \gamma_2}]) \to H^*(M_{\gamma_1, \gamma_2} / M_{\gamma_1, \gamma_2}) \to \cdots, \]
we see that \( \text{Im}(\phi_1) \) goes to 0 when we restrict it to
\[ H^{*+2c_1}([M_{U}/G_{Y_1,Y_2}\{M_{Y_1,Y_2}/G_{Y_1,Y_2}\}]. \]

Since \([M_{U}/G_{Y_1,Y_2}\{M_{Y_1,Y_2}/G_{Y_1,Y_2}\}] \text{ is contained in } [M_{U}/G_{Y_1,Y_2}\{M_{Y_1,Y_2}/G_{Y_1,Y_2}\}], \text{ Im(}\phi_1) \text{ vanishes when we restrict it further to} \]
\[ H^{*+2c_1}(M_{U}/G_{Y_1,Y_2}). \]

Now applying \(\phi_2\), we see that \(\text{Im}(\phi_2 \circ \phi_1)\) vanishes when we restrict it to \(H^{*+2c_1+2c_2}(M_{U}/G_{Y_1,Y_2})\). \(\square\)

We need some preliminary results before proceeding to the proof of the main theorem. Given an element \(t = (t_{ij})_{i \in I} \in t_{\gamma}^{0,\text{gen}}\), let \(O\) be its orbit under the action of \(PG_{\gamma}\) by conjugation. Recall that Crawley-Boevey [3] has identified the geometric quotient \(\mu^{-1}(O)/PG_{\gamma}\) with a quiver variety; let \(\tilde{Q}\) be the quiver obtained from \(Q\) by attaching to each vertex \(i \in I\) a leg of length \(\gamma_i - 1\). More precisely, vertices of \(\tilde{Q}\) are labeled \([i,j], i \in I, j = 0, \cdots, \gamma_i - 1\), and we identify \([i,0]\) with \(i\). Besides the arrows in \(Q\), the new arrows in \(\tilde{Q}\) are \([i,j] = [i,j + 1]\) for each \(i \in I\). \(j = 0, \cdots, \gamma_i - 2\). The new dimension vector of \(\tilde{Q}\) is defined to be \(\gamma_{i,j} = \gamma_i - j\). Again, we have the moment map \(\tilde{\mu} : M_{\tilde{Q}}^0 \to \gamma_0^0\), where \(M_{\tilde{Q}}^0\) is the space of representations of \(\tilde{Q}\) of dimension vector \(\gamma\). For each \(i \in I\), let \(t_{i,1}, \cdots, t_{i,\gamma_i}\) be the eigenvalues of \(t_i\). Define \(\gamma = (\gamma_{i,j}) \in \gamma_0^0\) to be
\[ \gamma_{i,0} = -t_{i,1}, \]
\[ \gamma_{i,j} = t_{i,j} - t_{i,j+1}, \quad j = 1, \cdots, \gamma_i - 1. \]

Notice that \(\vec{\gamma} \cdot \lambda = 0\). Now the result of Crawley-Boevey [3] states that
\[ \mu^{-1}(O)/PG_{\gamma} \cong \tilde{\mu}^{-1}(\lambda)/PG_{\gamma}. \]

Moreover, according to [6], corollary 1.6 (iv), \(\Omega(\gamma')(q)\) is non-zero if and only if \(\vec{\gamma}\) is a positive root of \(Q'\); here we write \(\tilde{Q}\) as the double of another quiver \(Q'\).

**Lemma 2.2.** The morphism \(f : U_{\gamma}^0/PG_{\gamma} \to \tilde{t}_{\gamma}^{0,\text{gen}}/W_{\gamma}\) makes \(U_{\gamma}^0/PG_{\gamma}\) a fiber bundle over \(t_{\gamma}^{0,\text{gen}}/W_{\gamma}\). Moreover, the sheaf \(R^if_{\gamma}^*Q\) is constant on the étale neighbourhood \(t_{\gamma}^{0,\text{gen}}/W_{\gamma}\) of \(t_{\gamma}^{0,\text{gen}}/W_{\gamma}\).

**Proof.** This is basically Lemma 48 of [9], with one difference. As in the proof of [6] Theorem 2.3, Maffei works with quiver without loops, but his proof carries over in our case. In his proof, the important point is the surjectivity of \(f\) (or rather the hyper-K"ahler moment map on the generic locus, but this can be reduced to \(f\) by hyper-K"ahler rotation). According to [2], Theorem 4.4, this is fulfilled in our situation, since \(\mu^{-1}(O)/PG_{\gamma} \cong \tilde{\mu}^{-1}(\lambda)/PG_{\gamma}\) and \(\vec{\gamma} \cdot \lambda = 0\), taking into account that \(\vec{\gamma}\) is a positive root of \(Q'\). \(\square\)

**Lemma 2.3** (Crawley-Boevey–van den Bergh). The smooth quasi-projective algebraic variety \(\mu^{-1}(O)/PG_{\gamma}\) has pure mixed Hodge structure.

**Proof.** This is a corollary of [4], §2.4, taking into account isomorphism (2). \(\square\)

Now we can prove the first part of Theorem 1.1.

**Theorem 2.1.** The pure part of \(H^*(U_{\gamma}^0/PG_{\gamma})\) is equal to \(H^*(\mu^{-1}(O)/PG_{\gamma})^W\gamma\).

**Proof.** Consider the fiber bundle \(\tilde{f} : U_{\gamma}^0/PG_{\gamma} \to \tilde{t}_{\gamma}^{0,\text{gen}}/W_{\gamma}\). By Lemma 2.2, the sheaf \(R^if_{\gamma}^*Q\) is constant on the étale neighbourhood \(t_{\gamma}^{0,\text{gen}}/W_{\gamma}\) of \(t_{\gamma}^{0,\text{gen}}/W_{\gamma}\), so we get the Hochschild–Serre spectral sequence,
\[ H_{\gamma}^{p,q}(U_{\gamma}/PG_{\gamma}) \implies H^{p+q}(U_{\gamma}/PG_{\gamma}). \]

Since \(W_{\gamma}\) is a finite group, we have \(E_2^{p,q} = 0\) for \(p \neq 0\). So the spectral sequence degenerates, and we get
\[ H^q(U_{\gamma}/PG_{\gamma}) = (H^q(t_{\gamma}^{0,\text{gen}}, Rf_{\gamma}^*Q)^W_{\gamma} = \left( \bigoplus_{q_1+q_2=q} H^{q_1}(t_{\gamma}^{0,\text{gen}}) \otimes H^{q_2}(\mu^{-1}(O)/PG_{\gamma}) \right)^W_{\gamma}. \]

Since \(t_{\gamma}^{0,\text{gen}}\) is the complement of unions of sufficiently many hyperplanes in the vector space \(t_{\gamma}^0\), one proves easily by induction on the number of hyperplanes that the mixed Hodge structure of \(H^i(t_{\gamma}^{0,\text{gen}})\) is not pure if \(i \neq 0\). So the pure part of \(H^q(U_{\gamma}/PG_{\gamma})\) is exactly \(H^q(\mu^{-1}(O)/PG_{\gamma})^W_{\gamma}\). \(\square\)
To prove the second result in the main theorem, we need some facts from algebraic stacks. We refer the reader to Olsson–Lazslo [10,11] and Sun [12] for the proofs. Although they work over the finite fields, their results apply in our situation, since the quiver varieties are in fact Z-schemes. The moduli stack \( \{M_{\gamma}/PG_{\gamma}\} \) is a smooth Artin stack over \( \mathcal{C} \) with dimension \( d_{\gamma} = \dim(M_{\gamma}) - \dim(PG_{\gamma}) \). It has dualizing complex \( Q(d_{\gamma})[2d_{\gamma}] \), and we have the Poincaré duality, which is a perfect non-degenerate bilinear pairing

\[
H^{i}(\{M_{\gamma}/PG_{\gamma}\}) \times H_{c}^{2d_{\gamma}-i}(\{M_{\gamma}/PG_{\gamma}\}) \to Q(d_{\gamma}).
\]

Using the fibration \( \{M_{\gamma}/PG_{\gamma}\} \to \{pt/PG_{\gamma}\} \), we have \( H^{i}(\{M_{\gamma}/PG_{\gamma}\}) = H^{i}_{PG_{\gamma}}(pt) \) is pure of weight \( i \), the groups \( H_{c}^{2d_{\gamma}-i}(\{M_{\gamma}/PG_{\gamma}\}) \) are all pure of weight \( 2d_{\gamma} - i \).

Furthermore, let \( Z_{\gamma} = M_{\gamma} \setminus U_{\gamma} \), then \( H^{i}_{c}(\{Z_{\gamma}/PG_{\gamma}\}) \) is of weight less than or equal to \( i \). This is essentially [13]. More precisely, as in [1], let \( E \to B_{n} \in \mathbb{N} \) be an injective system of finite dimensional \( n \)-acyclic approximation to the universal \( PG_{\gamma} \)-torsor \( E \to B \), then

\[
H^{i}_{c}(\{Z_{\gamma}/PG_{\gamma}\}) = \lim_{n \to \infty} H^{i+2 \dim(E_{n})}(Z_{\gamma} \times_{PG_{\gamma}} E_{n})(- \dim(E_{n})).
\]

Now it suffices to apply [13] to the right-hand side.

**Proof of the second part of Theorem 1.1.** We have the long exact sequence:

\[
\cdots \to H_{c}^{i-1}(\{Z_{\gamma}/PG_{\gamma}\}) \to H_{c}^{i}(\{U/PG_{\gamma}\}) \to H_{c}^{i}(\{M_{\gamma}/PG_{\gamma}\}) \to H_{c}^{i}(\{Z_{\gamma}/PG_{\gamma}\}) \to \cdots.
\]

(3)

Since \( H_{c}^{i-1}(\{Z_{\gamma}/PG_{\gamma}\}) \) is of weight less than or equal to \( i - 1 \), the pure part of \( H_{c}^{i}(\{U/PG_{\gamma}\}) \) injects into \( H_{c}^{i}(\{M_{\gamma}/PG_{\gamma}\}) \). Taking Poincaré duality, we have that \( H^{2d_{\gamma}-i}(\{M_{\gamma}/PG_{\gamma}\}) \) maps onto the pure part of \( H^{2d_{\gamma}-i}(\{U/PG_{\gamma}\}) \). By Theorem 2.1, we have surjective morphism

\[
H^{i}(\{M_{\gamma}/PG_{\gamma}\}) \to H^{j}(\mu^{-1}(O)/PG_{\gamma})^{W_{\gamma}}, \quad \forall j.
\]

By the definition of \( t^{0,\text{gen}}_{\gamma} \), we find easily that \( U_{\gamma} \subseteq M_{\gamma}^{\text{st}} \). So the above map factorise by

\[
H^{j}(\{M_{\gamma}/PG_{\gamma}\}) \to H^{j}(M_{\gamma}^{\text{st}}/PG_{\gamma}) \to H^{j}(\mu^{-1}(O)/PG_{\gamma})^{W_{\gamma}}.
\]

(4)

By Lemma 2.1, the first arrow has the same image as its restriction to \( \{\gamma:j+\chi_{\gamma}(\gamma,\gamma)\} \), so we get a surjective morphism

\[
\{\gamma:j+\chi_{\gamma}(\gamma,\gamma)\} \to H^{j}(\mu^{-1}(O)/PG_{\gamma})^{W_{\gamma}}.
\]

By the result of [6] recalled in the introduction, they have the same dimension, so they are isomorphic. \( \square \)

Similar arguments can be used to show the following variant of the geometric construction.

**Theorem 2.2.** The restriction

\[
H^{*}(\{M_{\gamma}/PG_{\gamma}\}) \to H^{*}(\{M_{\gamma}^{\text{st}}/PG_{\gamma}\})
\]

induces an isomorphism

\[
\{\gamma:j+\chi_{\gamma}(\gamma,\gamma)\} \to \text{PH}^{*}(\{M_{\gamma}^{\text{st}}/PG_{\gamma}\})
\]

where \( \text{PH}^{*}(\{M_{\gamma}^{\text{st}}/PG_{\gamma}\}) \) is the pure part of \( H^{*}(\{M_{\gamma}^{\text{st}}/PG_{\gamma}\}) \).

**Proof.** The proof is almost the same as that of the main theorem, we indicate only the differences. Using an exact sequence as (3), with the pair \( (U_{\gamma}, Z_{\gamma}) \) replaced by \( (M_{\gamma}^{\text{st}}, M_{\gamma}/M_{\gamma}^{\text{st}}) \), we can show that the restriction \( H^{*}(\{M_{\gamma}/PG_{\gamma}\}) \to H^{*}(\{M_{\gamma}^{\text{st}}/PG_{\gamma}\}) \) factors through and is surjective onto \( \text{PH}^{*}(\{M_{\gamma}^{\text{st}}/PG_{\gamma}\}) \). Again by Lemma 2.1, we get the surjection

\[
\{\gamma:j+\chi_{\gamma}(\gamma,\gamma)\} \to \text{PH}^{*}(\{M_{\gamma}^{\text{st}}/PG_{\gamma}\})
\]

Now observe that the second morphism in factorisation (4) has the same image as that of

\[
\text{PH}^{j}(\{M_{\gamma}^{\text{st}}/PG_{\gamma}\}) \to H^{j}(\mu^{-1}(O)/PG_{\gamma})^{W_{\gamma}},
\]

since the morphism preserves the weights of the cohomological groups and \( H^{j}(\mu^{-1}(O)/PG_{\gamma})^{W_{\gamma}} \) is pure of weight \( j \). So factorisation (4) becomes:

\[
\{\gamma:j+\chi_{\gamma}(\gamma,\gamma)\} \to \text{PH}^{j}(\{M_{\gamma}^{\text{st}}/PG_{\gamma}\}) \to H^{j}(\mu^{-1}(O)/PG_{\gamma})^{W_{\gamma}}.
\]

(5)

Now that the composition is an isomorphism by our main theorem, all the arrows in (5) are isomorphisms. \( \square \)
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References