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On a projectively invariant distance on Finsler spaces of constant negative Ricci scalar

Sur une distance projectivement invariante dans les espaces d'Einstein–Finsler

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ABSTRACT

In this work, an intrinsic projectively invariant distance is used to establish a new approach to the study of projective geometry in a Finsler space. It is shown that the projectively invariant distance previously defined is a constant multiple of the Finsler distance when the manifold in question is both forward and backward complete. As a consequence, two projectively related complete Einstein Finsler spaces with constant negative scalar curvature are homothetic. Evidently, this will be true for Finsler spaces of constant flag curvature as well.

RÉSUMÉ

Dans ce travail, une distance intrinsèque projectivement invariante est utilisée pour établir une nouvelle approche en vue de l'étude de la géométrie projective dans les espaces de Finsler. Il est démontré que la distance projectivement invariante définie précédemment est un multiple constant de la distance finslérienne dans le cas où celle-ci est complète (à la fois en avant et en arrière). Par conséquent, deux espaces d'Einstein–Finsler complets à courbure scalaire constante négative sont homothétiques. Évidemment, ceci sera vrai aussi pour les espaces de Finsler à courbure sectionelle constante.

1. Introduction

Two regular metrics on a manifold are said to be pointwise projectively related if they have the same geodesics as the point sets. Two regular metric spaces are said to be projectively related if there is a diffeomorphism between them such that the pull-back of one metric is pointwise projective to the other. Let γ be a geodesic of a metric space. In general, the parameter t of $\gamma(t)$ does not remain invariant under the projective changes. There is a unique parameter up to linear fractional

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transformations which is projectively invariant. This parameter is referred to, in the literature, as *projective parameter*. See [4,10] for a survey. The projective parameter together with the Poincaré metric is used to establish a projectively invariant pseudo-distance in Finsler spaces. Next a comparison theorem on Ricci curvatures shows that this pseudo-distance is a distance. The Ricci tensor was introduced in Riemannian spaces in 1904 by G. Ricci and nine years later, it was used to formulate Einstein's theory of gravitation [5]. In the present work, we use the notion of Ricci curvature introduced by Akbar–Zadeh, cf. [1]. Hence a Finsler metric is said to be Einstein if the Ricci scalar Ric is a function of x alone. Equivalently, $Ric_{ij} = Ric(x)g_{ij}$.

Without pretending to be exhaustive, we bring a few results related to our approach to Einstein–Finsler spaces. If M is simply connected endowed with a complete metric connection such that the symmetric part of the Ricci curvature of the associated symmetric connection is of Einstein type, that is, $R_{(ij)} = cg_{ij}$, where c is a positive constant and, if M admits a projective group leaving invariant the trace of torsion, then M is homeomorphic to an sphere, cf. [2]. In [11], Z. Shen found out that two pointwise projectively equivalent Einstein–Finsler metrics F and $\bar{F}$ on an n-dimensional compact manifold M have Einstein constants of the same sign. In addition, if two pointwise projectively related Einstein metrics are complete with negative Einstein constants, then one of them is a multiple of the other. Later in a joint work, he proved that if two projectively related Riemannian metrics g and $\bar{g}$ on a manifold M have Ricci curvatures satisfying $\bar{Ric} \le Ric$ and g is complete, then the projective change is affine [7]. Recently, G. Yang generalized this comparison on Ricci curvatures of Finsler spaces and got some interesting results about the length of geodesics and the completeness of the space [12]. Here, inspired by Kobayashi's work [8], the projectively invariant distance in complete Einstein spaces is studied and it is proved that the intrinsic distance is a constant multiple of the Finslerian distance. Consequently the topology generated by the intrinsic distance coincide with that of Finslerian distance and, in a new approach, we find out the known fact that two projectively related complete Einstein–Finsler spaces with constant negative Ricci scalar are homothetic, cf. [11].

2. Preliminaries

Let M be an n-dimensional C^∞ connected manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM := \bigcup_{x\in M} T_xM$ the bundle of tangent spaces. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. The natural projection $\pi : TM \to M$, is given by $\pi(x, y) := x$. The pull-back tangent bundle π^*TM is a vector bundle over the slit tangent bundle $TM_0 := TM\backslash\{0\}$, for which the fiber π_v^*TM at $v \in TM_0$ is just T_xM, where $\pi(v) = x$.

A (globally defined) Finsler structure on M is a function $F : TM \to [0,\infty)$ with the properties; (I) regularity: F is C^∞ on the entire slit tangent bundle TM_0; (II) positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$; (III) strong convexity: The $n \times n$ Hessian matrix $(g_{ij}) := ([\frac{1}{2}F^2]_{y^iy^j})$ is positive-definite at every point of TM_0. For any $y \in T_xM_0$, the Hessian $g_{ij}(y)$ induces an inner product g_y in T_xM by $g_y(u, v) := g_{ij}(y)u^iv^j$. Let $\gamma : [a,b] \to M$ be a piecewise C^∞ curve on (M, F) with the velocity $\frac{d\gamma}{dt} = \frac{d\gamma^i}{dt}\frac{\partial}{\partial x^i} \in T_{\gamma(t)}M$. The arc-length parameter of γ is given by $s(t) = \int_{t_0}^t F(\gamma, \frac{d\gamma}{dr})dr$, and the integral length is denoted by $L(\gamma) := \int_a^b F(\gamma, \frac{d\gamma}{dt})dt$. For every $x_0, x_1 \in M$, denote by $\Gamma(x_0, x_1)$ the collection of all piecewise C^∞ curves $\gamma : [a,b] \to M$ with $\gamma(a) = x_0$ and $\gamma(b) = x_1$, and define a map $d_F : M \times M \to [0,\infty)$ by $d_F(x_0, x_1) := \inf L(\alpha)$, where $\alpha \in \Gamma(x_0, x_1)$. It can be shown that d_F satisfies the first two axioms of a metric space. Namely, (I) $d_F(x_0, x_1) \ge 0$, where equality holds if and only if $x_0 = x_1$; (II) $d_F(x_0, x_1) \le d_F(x_0, x_1) + d_F(x_1, x_2)$.

We should remark that the distance function d_F on a Finsler space does not have the symmetry property. If the Finsler structure F is absolutely homogeneous, that is $F(x, \lambda y) = |\lambda|F(x, y)$ for $\lambda \in \mathbb{R}$, then one also has the third axiom of a metric space, (III) $d(x_0, x_1) = d(x_1, x_0)$. The manifold topology coincides with that generated by the forward metric balls, $B_p^+(r) := \{x \in M : d_F(p, x) < r\}$. The latter assertion is also true for backward metric balls, $B_p^-(r)$, cf. [3]. Every Finsler metric F induces a spray $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ on TM, where $G^i(x, y) := \frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\}$. $\mathbf{G}$ is a globally defined vector field on TM. The projection of a flow line of $\mathbf{G}$ is called a geodesic on M. A curve $\gamma(t)$ on M is a geodesic if and only if in local coordinate it satisfies $\frac{d^2x^i}{ds^2} + 2G^i(x(s), \frac{dx}{ds}) = 0$, where s is the arc-length parameter. F is said to be forward (resp. backward) geodesically complete if any geodesic on an open interval (a,b) can be extended to a geodesic on (a,∞) (resp. $(-\infty, b)$). F is said to be complete if it is forward and backward complete. For a vector $y \in T_xM_0$, the Riemann curvature $\mathbf{R}_y : T_xM \to T_xM$ is defined by $\mathbf{R}_y(u) = R^i_k u^k\frac{\partial}{\partial x^i}$, where $R^i_k(y) := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2G^i}{\partial y^k\partial x^j}y^j + 2G^j\frac{\partial^2G^i}{\partial y^k\partial y^j} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$.

For a two-dimensional plane $P \subset T_pM$ and a non-zero vector $y \in T_pM$, the flag curvature $\mathbf{K}(P, y)$ is defined by $\mathbf{K}(P, y) := \frac{g_y(u, \mathbf{R}_y(u))}{g_y(y,y)g_y(u,u) - g_y(y,u)^2}$, where $P = \mathrm{span}\{y, u\}$. F is said to be of *scalar curvature* $\mathbf{K} = \lambda(y)$ if for any $y \in T_pM$, the flag curvature $\mathbf{K}(P, y) = \lambda(y)$ is independent of P containing $y \in T_pM$. It is equivalent to the following system in a local coordinate system (x^i, y^i) on TM:

$$R^i_k = \lambda F^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}. \tag{1}$$

If λ is a constant, then F is said to be of *constant curvature*. The Ricci scalar of F is a positive zero homogeneous function in y given by $Ric := \frac{1}{F^2}R^i_i$. This is equivalent to say that $Ric(x, y)$ depends on the direction of the flag pole y but not on its length. The Ricci tensor of a Finsler metric F is defined by $Ric_{ij} := \{\frac{1}{2}R^k_k\}_{y^iy^j}$, cf. [1]. If (M, F) is a Finsler space with constant flag curvature λ, (1) leads to:

$$Ric = (n-1)\lambda, \qquad Ric_{ij} = (n-1)\lambda g_{ij}. \tag{2}$$

A Finsler metric is said to be an *Einstein metric* if the Ricci scalar function is a function of x alone, equivalently $Ric_{ij} = Ric(x)g_{ij}$.

2.1. Projective parameter and Schwarzian derivative

A Finsler space (M, F) is said to be *projective* to another Finsler space $(M, \bar{F})$ as a set of points, if and only if there exists a one-positive homogeneous scalar field $P(x, y)$ on TM satisfying $\bar{G}^i(x, y) = G^i(x, y) + P(x, y)y^i$. The scalar field $P(x, y)$ is called the *projective factor* of the projective change under consideration. In general, the parameter t of a geodesic does not remain invariant under projective change of metrics. It is well known that there is a unique parameter up to the linear fractional transformations that is projectively invariant. This parameter is referred to, in the literature, as a *projective parameter*. The projective parameter, for a geodesic γ, is given by $\{\pi, s\} = \frac{2}{n-1} Ric_{jk} \frac{dx^j}{ds}\frac{dx^k}{ds}$, where the operator $\{.,.\}$ is the *Schwarzian derivative* defined for a C^∞ real function f on $\mathbb{R}$, and for $t \in \mathbb{R}$ by $\{f, t\} = \frac{f'''}{f'} - 3/2(\frac{f''}{f'})^2$, where f', f'', f''' are first, second, and third derivatives of f with respect to t. It is invariant under all linear fractional transformations, namely $\{\frac{af+b}{cf+d}, t\} = \{f, t\}$, where $ad - bc \neq 0$. A geodesic $\gamma : I \to M$ is said to be projective if its natural parameter on I is a projective parameter.

3. Projectively invariant intrinsic distance in complete Einstein spaces

Consider the open interval $I = (-1, 1)$ with Poincaré metric $ds_I^2 = \frac{4du^2}{(1-u^2)^2}$. The Poincaré distance between two points u_0 and u_1 in I is given by

$$\rho(a, b) = \left| \ln \frac{(1-a)(1+b)}{(1-b)(1+a)} \right| = \left| \ln(1, -1; u_0, u_1) \right|, \tag{3}$$

where $(1, -1; u_0, u_1)$ denotes the cross-ratio between u_0 and u_1 with respect to 1 and -1, cf., [9]. Now, we are in a position to define the pseudo-distance d_M on a Finsler space (M, F). Given any two points $x, y \in M$, we choose a chain α of geodesic segments consisting of (I) a chain of points $x = x_0, x_1, ..., x_k = y$ on M, (II) pairs of points $a_1, b_1, ..., a_k, b_k$ in I, (III) projective maps $f_1, ..., f_k$, $f_i : I \to M$, such that $f_i(a_i) = x_{i-1}$, $f_i(b_i) = x_i$, $i = 1, ..., k$. The length $L(\alpha)$ of the chain α is defined to be $L(\alpha) = \Sigma_i \rho(a_i, b_i)$. The pseudo-distance $d_M(x, y)$ is defined by $d_M(x, y) = \inf L(\alpha)$, where the infimum is taken over all chains α from x to y. It is well known d_M remains invariant under the projective change of metrics and we have the following lemmas, cf., [10].

Lemma 3.1. *(I) Let the geodesic $f : I \to M$ be a projective map, then $\rho(a, b) \geq d_M(f(a), f(b))$ for any $a, b \in I$. (II) Let δ_M be any pseudo-distance on M with the property $\rho(a, b) \geq \delta_M(f(a), f(b))$ for any $a, b \in I$, and for all projective maps $f : I \to M$, then $\delta_M(x, y) \leq d_M(x, y)$ for any $x, y \in M$.*

Lemma 3.2. *Let (M, F) be a Finsler space for which the Ricci tensor satisfies $Ric_{ij} \leq -c^2 g_{ij}$ as matrices, for a positive constant c. Let $d_F(.,.)$ be the distance induced by F, then for every projective map $f : I \to M$, d_F is bounded below by the Poincaré distance ρ, that is, $\rho(a, b) \geq \frac{2c}{\sqrt{n-1}} d_F(f(a), f(b))$, $\forall a, b \in I$.*

Proposition 3.1. *Let (M, F) be a Finsler space for which the Ricci tensor satisfies $Ric_{ij} \leq -c^2 g_{ij}$ as matrices, for a positive constant c. Then the pseudo-distance d_M is a distance.*

Following the procedure described above by collecting properties of the projectively invariant distance d_M, we are in a position to prove the following theorem.

Theorem 3.3. *Let (M, F) be a complete Einstein Finsler space with*

$$Ric_{ij} = -c^2 g_{ij}, \tag{4}$$

where c is a positive constant. Then d_M the projectively invariant distance is proportional to the Finslerian distance d_F, that is:

$$d_M(x, y) = \frac{2c}{\sqrt{n-1}} d_F(x, y). \tag{5}$$

It would be noteworthy to remark that the constant c^2 in (4) is of the order of $(n-1)$ because the Ricci scalar is the sum of $(n-1)$ flag curvatures, thus c is of the order of $\sqrt{n-1}$. This would make it plausible that in (5), the multiplier $2c/\sqrt{n-1}$ is asymptotically independent of n.

Proof. By means of the second part of Lemma 3.1 and Lemma 3.2, we have $d_F(x,y)\frac{2c}{\sqrt{n-1}k} \le d_M(x,y)$. To prove the assertion, it remains to show the converse. Given any two points x, y on M, we take a minimizing geodesic $x(s)$ on M parameterized by an arc length s satisfying $x = x(0)$ and $y = x(a)$, where a is the Finslerian distance from x to y. A projective parameter π for this geodesic is defined to be a solution to the differential equation $\{\pi, s\} = \frac{2}{n-1} Ric_{jk} \frac{dx^j}{ds}\frac{dx^k}{ds}$.

Let us consider the assumption (4) which leads to $\{\pi, s\} = \frac{-2c^2}{n-1} g_{jk} \frac{dx^j}{ds}\frac{dx^k}{ds}$. For all curves parameterized by arc length $g_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 1$, therefore $\{\pi, s\} = \frac{-2c^2}{n-1}$. The general solution to the latter equation is given by

$$\pi(s) = \frac{\alpha\, e^{js} + \beta\, e^{-js}}{\gamma\, e^{js} + \delta\, e^{-js}}, \tag{6}$$

where $\alpha\delta - \beta\gamma \neq 0$, and $j = \frac{c}{\sqrt{n-1}}$. According to the first part of Lemma 3.1, the Poincaré distance ρ between the points 0 and $\pi(a)$ in I, satisfies

$$\rho\big(0, \pi(a)\big) \ge d_M(x,y). \tag{7}$$

We consider the special solution to (6), that is $\pi(s) = \frac{e^{js} - e^{-js}}{e^{js} + e^{-js}}$. Thus $\pi(-\infty) = -1$, $\pi(0) = 0$ and $\pi(+\infty) = 1$. Plugging $\pi(s)$ into (3) leads to $\rho(0, \pi(a)) = |\ln(\frac{1+\pi(a)}{1-\pi(a)})| = 2ja = \frac{2ca}{\sqrt{n-1}}$. Therefore $d_M \le \frac{2ca}{\sqrt{n-1}} = \frac{2c}{\sqrt{n-1}} d_F(x,y)$. This completes the proof. $\square$

To present an explicit **counterexample** to Theorem 3.3, when completeness is violated, we consider the Finslerian–Poincaré disc in the following example.

Example 1. Let M be the Euclidean open disc with radius 2 and centered at $(0,0)$ in $\mathbb{R}^2$ and F the Finslerian–Poincaré metric defined by $F(V) := \sqrt{\tilde{a}(V,V)} + \tilde{b}(V)$, for $V \in T_xM$, where in polar coordinates $\tilde{a} := \frac{1}{(1-\frac{r^2}{4})^2}[dr \otimes dr + r^2 d\theta \otimes d\theta]$ and $\tilde{b} := d[\ln \frac{(4+r^2)}{(4-r^2)}] = \frac{r}{(1+r^2/4)(1-r^2/4)} dr$. It is well known that **the Finslerian Poincaré metric is not backward complete**. It is forward complete and of constant curvature $K = -1/4$, cf., [3], pp. 333–342. Geodesics of Poincaré disc have the following trajectories:

1) Euclidean circular arcs that intersect the boundary of the Poincaré disc at Euclidean right angles, where none of these can pass through the origin, 2) Euclidean straight rays that emanate from the origin, 3) Euclidean straight rays that aim towards the origin. We have $d_F(P, O) = \ln[\frac{(2+\epsilon)^2}{(4+\epsilon^2)}]$. The ray that aims towards the origin is globally the shortest among all curves from P to O. Let us consider the point $P = (0,1)$. Assume that "s" is the arc-length parameter for the geodesic γ emanating from the origin to P. The projective parameter for this geodesic is defined to be a solution to the ODE, $\{\pi, s\} = \frac{-2c^2}{n-1} g_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}$. Since M has constant curvature $-1/4$, this ODE reads $\{\pi, s\} = \frac{-2c^2}{n-1}$ and its general solution is given by $\pi(s) = \frac{\alpha e^{2ks} + 1}{\beta e^{2ks} + \delta}$, where, $k = c/\sqrt{n-1}$ and $\alpha\delta - \beta \neq 0$. Here by our assumption $c = 1/2$. We note that $L(\gamma) = d_F(O, P) = \ln 5$, $\gamma(0) = O$ and $\gamma(\ln 5) = P$. Considering projective parameter, we have $\gamma(\pi(0)) = O$ and $\gamma(\pi(\ln 5)) = P$. Next, we show that $d_M(O, P) \ge \ln(5+\epsilon) > d_F(O, P)$, for some $\epsilon > 0$. Without loss of generality, we assume $\alpha\delta - \beta > 0$. In this case, π is strictly increasing. We show that there is no projective parameter that satisfies the following statements. i) $\pi(-\ln 2) \le -1$, ii) $\lim_{s\to+\infty} \pi(s) \ge 1$, iii) $\rho(\pi(0), \pi(\ln 5)) = \ln(5+\epsilon)$, $\forall \epsilon < 16/5$. The first two statements are necessary to guarantee that π is defined on $(-1,1)$. We proceed by contradiction. Let $0 < \epsilon < 16/5$ and assume that we have a projective parameter, $\pi(s) = \frac{\alpha e^s + 1}{\beta e^s + \delta}$, which satisfies the statements i)–iii). We have $-1 < \pi(0) < 1$ and $-1 < \pi(\ln 5) < 1$ which read:

$$\frac{\beta + \delta + \alpha + 1}{\beta + \delta} > 0, \qquad \frac{\beta + \delta - \alpha - 1}{\beta + \delta} > 0, \tag{8}$$

$$\frac{5\beta + \delta + 5\alpha + 1}{5\beta + \delta} > 0, \qquad \frac{5\beta + \delta - 5\alpha - 1}{5\beta + \delta} > 0. \tag{9}$$

Also i) and ii) read

$$\frac{\alpha + 2}{\beta + 2\delta} \le -1, \qquad \frac{\alpha}{\beta} \ge 1. \tag{10}$$

The case $\lim_{s\to+\infty} \pi(s) = +\infty$ happens when $\beta = 0$ and it is considered in (10). Since $\frac{\alpha}{\beta} \ge 1$, we should study two cases $\alpha \ge \beta > 0$ and $\alpha \le \beta < 0$. First, we assume that $\alpha \ge \beta > 0$. Since $\alpha\delta > \beta$, δ should be positive. Therefore $\frac{\alpha+2}{\beta+2\delta} > 0$ which contradicts (10). We consider $\alpha \le \beta < 0$ henceforth. Now let study δ in the following four cases.

Case 1) Assume that $\delta \le -1$. From (10), we have $\alpha + 2 \ge -(\beta + 2\delta)$ and $\alpha + \beta \ge -2(1+\delta)$. The right hand side of the inequality is positive and it is a contradiction.

Case 2) Assume that $\delta \ge 1$. We have $\alpha\delta \le \beta$ which is a contradiction.

Case 3) Assume that $-1 < \delta \le 0$. iii) reads $\rho(\pi(0), \pi(\ln 5)) = |\ln \frac{(1-\pi(0))(1+\pi(\ln 5))}{(1+\pi(0))(1-\pi(\ln 5))}| = \ln \frac{(1-\pi(0))(1+\pi(\ln 5))}{(1+\pi(0))(1-\pi(\ln 5))} = \ln(5+\epsilon)$. The second equality holds because π is strictly increasing. We have $(\frac{\beta-\alpha+\delta+1}{\beta+\delta+\alpha+1})(\frac{5\beta+\delta+5\alpha+1}{5\beta+\delta-5\alpha-1}) = 5+\epsilon$. Therefore, $(20+5\epsilon)(\beta^2 - \alpha^2) + (24+5\epsilon)(\beta-\alpha)(\delta+1) + \epsilon(\beta+\alpha)(\delta-1) + (4+\epsilon)(\delta^2-1) = 0$. Rewriting it, we have

$$(\beta-\alpha)\big[(20+5\epsilon)(\beta+\alpha) + (24+5\epsilon)(\delta+1)\big] = (1-\delta)\big[4(1+\delta) + \epsilon(\beta+\alpha+\delta+1)\big]. \tag{11}$$

According to (10), $\beta+\delta+\alpha+1 > -(1+\delta)$ and $4(1+\delta)+\epsilon(\alpha+\beta+\delta+1) > (4-\epsilon)(1+\delta)$. Therefore, $4(1+\delta)+\epsilon(\alpha+\beta+\delta+1) > 0$. Moreover, according to (11), $(20+5\epsilon)(\beta+\alpha)+(24+5\epsilon)(\delta+1) > 0$. From (8), we have $\beta+\alpha < -1-\delta$ and $0 < (20+5\epsilon)(\beta+\alpha)+(24+5\epsilon)(\delta+1) < (20+5\epsilon)(-\delta-1)+(24+5\epsilon)(\delta+1) = 4(1+\delta)$. Eq. (11) and the last equation read $\beta-\alpha > \frac{(1-\delta)[4(1+\delta)+\epsilon(\alpha+\beta+\delta+1)]}{4(1+\delta)} = (1-\delta)[1+\frac{\epsilon}{4}(1+\frac{\beta+\alpha}{1+\delta})]$. According to (8) and (10), we have $-2-2\delta \le \alpha+\beta \le -1-\delta$. Therefore $-2 \le \frac{\alpha+\beta}{1+\delta} < -1$ and $1-\frac{\epsilon}{4} \le 1+\frac{\epsilon}{4}(1+\frac{\beta+\alpha}{1+\delta}) < 1$. $\epsilon < \frac{16}{5}$. Thus we have $\epsilon(1+\frac{\beta+\alpha}{1+\delta}) \ge -\epsilon > \frac{-16}{5}$, $1+\frac{\epsilon}{4}(1+\frac{\beta+\alpha}{1+\delta}) > \frac{1}{5}$ and $\beta-\delta > \frac{1}{5}(1-\delta)$. The latter inequality contradicts (9).

Case 4) Assume that $0 < \delta < 1$. Here we study two cases $\beta+\delta > 0$ and $\beta+\delta < 0$. Assume that $\beta+\delta > 0$. According to (8) $\beta+\alpha > -1-\delta$. Adding the term $2+2\delta$ to both side, we have $\alpha+2+\beta+2\delta > \delta+1 > 0$. Since $\beta+2\delta > 0$, it contradicts (10). Now assume that $\beta+\delta < 0$. Therefore $5\beta+\delta < 0$. If $\beta+2\delta < 0$, we just discussed the case. Finally, assume that $\beta+2\delta > 0$. Since $\alpha\delta > \beta$, we have $\alpha\delta+2\delta > \beta+2\delta > 0$. Here we have $\delta(\alpha+2) > 0$ and this contradicts (10). Now, from 1)–4) we have $\rho(\pi(0), \pi(\ln 5)) \ge \ln(5+\epsilon)$. By definition we have $d_M(O,P) \ge \ln(5+\epsilon) > d_F(O,P)$.

Two Finsler structures F and $\bar{F}$ are said to be homothetic if there is a constant λ such that $F = \lambda\bar{F}$. Let (M,F) be a Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one. Let $\sigma(t)$, $0 \le t < \epsilon$ be any short enough C^1 curve that emanates from p with initial velocity $v := v^i \frac{\partial}{\partial x^i}$.

One can show that $\sigma(t) = \exp_p(y_t)$, where y_t is a curve in T_pM that emanates from the origin with initial velocity v. Roughly speaking, the local coordinates x^i on M induce global coordinates y^i on T_pM. Therefore, we should have asserted that the initial velocity of y_t is $v^i \frac{\partial}{\partial x^i}$. However, this is a forgivable confusion on linear spaces, of which T_pM is one. We have $d_F(p,\sigma(t)) = F(p, y_t)$ and $v = \lim_{t\to 0^+} \frac{1}{t} y_t$. Using the continuity of F help us to deduce that $F(p,v) = \lim_{t\to 0^+} \frac{d_F(p,\sigma(t))}{t}$. In fact, this is a result due to the Busemann–Mayer theorem for positively homogeneous functions [3,6].

Now, let (M,F) and $(M,\bar{F})$ be two projectively related complete Finsler spaces with constant negative Ricci scalars. Due to Theorem 3.3, $\frac{2c}{\sqrt{n-1}} d_F(x,y) = d_M(x,y) = \frac{2\bar{c}}{\sqrt{n-1}} d_{\bar{F}}(x,y)$. Therefore, $d_F(x,y) = \lambda d_{\bar{F}}(x,y)$ for some λ. Considering the latter assertion we just made and Theorem 3.3, the following corollary is easily obtained.

Corollary 3.4. *Let (M,F) and $(M,\bar{F})$ be two complete Einstein–Finsler spaces with $Ric_{ij} = -c^2 g_{ij}$ and $\bar{Ric}_{ij} = -\bar{c}^2 \bar{g}_{ij}$, respectively, if F and $\bar{F}$ are projectively related, then they are homothetic.*

Moreover, if the flag curvature λ is constant then by means of Eq. (2), we have the following corollary.

Corollary 3.5. *Let (M,F) and $(M,\bar{F})$ be two complete Finsler spaces of constant negative flag curvature, if F and $\bar{F}$ are projectively related, then they are homothetic.*

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