Number theory

On small zeros of automorphic $L$-functions

Petits zéros des fonctions $L$ de formes automorphes

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ABSTRACT

In this paper, we first formulate the Weil explicit formula of prime number theory for cuspidal automorphic $L$-functions $L(s, \pi)$ of $GL_d$. Then, we prove some conditional results about the vanishing order at the central point of $L(s, \pi)$. This enables to yield an estimate for the height of the lowest zero of $L(s, \pi)$ on the critical line in terms of the analytic conductor.

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RÉSUMÉ

Dans cet article, nous formulons d’abord les formules explicites de Weil de la théorie des nombres premiers pour les fonctions $L$ de formes automorphes cuspidales $L(s, \pi)$ de $GL_d$. Ensuite, nous montrons des résultats conditionnels concernant l’ordre d’annulation de $L(s, \pi)$ au point $s = 1/2$, ce qui permet de donner une estimation de la hauteur du plus petit zéro de $L(s, \pi)$ sur la droite critique en termes de conducteur analytique.

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1. Introduction

Special values of $L$-functions often carry relevant arithmetic or geometric information on the objects that were used to define the $L$-functions. One is particularly interested in the vanishing or non-vanishing of various families of $L$-functions at $s = 1/2$ in the standard normalization.

In this paper, we give under the Riemann hypothesis some estimates for the order $n_\pi$ of an eventual zero of a cuspidal automorphic $L$-function $L(s, \pi)$ of $GL_d$ at the point $s = 1/2$ and for the height of the lowest zero of $L(s, \pi)$ on the critical line in terms of the analytic conductor. For this purpose, we shall first formulate Weil’s explicit formula in the context of cuspidal automorphic $L$-functions. Let $K$ be an algebraic number field of degree $n$, $O_K$ the ring of integers and $A_K$ the adele ring of $K$. Let $S_f$ and $S_\infty$ be the sets of all finite and infinite places of $K$, respectively. Write $S_\infty = S_\mathbb{R} \sqcup S_\mathbb{C}$, where $S_\mathbb{R}$ (resp. $S_\mathbb{C}$) is the set of all real (resp. complex) places of $K$ and put $r_1 = \# S_\mathbb{R}$ (resp. $r_2 = \# S_\mathbb{C}$). Let $\pi = \bigotimes_p \pi_p$.

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be an irreducible cuspidal automorphic representation of $GL_d(A_K)$. Then, from the general theory [3], we can define the $L$-function $L(s, \pi)$ by the Euler product:

$$L(s, \pi) = \prod_{v \in S_f} \prod_{j=1}^{d} (1 - \alpha_{v,j}(\pi) q_v^{-s})^{-1} = \sum_{n=1}^{\infty} \Lambda_\pi(n) n^{-s} \quad (\text{Re}(s) > 1),$$

where $q_v$ is the residue degree of the local field $K_v$ with $K_v$ being the $v$-adic completion of $K$ at $v$ and the complex number $\alpha_{v,j}(\pi)$ is determined by the local representation $\pi_v$ for each $v \in S_f$. From the Euler product expression of $L(s, \pi)$, we get

$$-\frac{L'}{L}(s, \pi) = \sum_{v \in S_f} \sum_{j=1}^{d} \Lambda_\pi(q_v^s),$$

where $\Lambda_\pi(q_v^s) := \log q_v \sum_{j=1}^{d} \alpha_{v,j}(\pi)^j$. Moreover, let $\Lambda(s, \pi)$ be the completed $L$-function defined by

$$\Lambda(s, \pi) = L_\infty(s, \pi) L(s, \pi),$$

where $L_\infty(s, \pi)$ is defined by

$$L_\infty(s, \pi) = \prod_{v \in S_\infty} \prod_{j=1}^{d} \Gamma_v(s + \mu_{v,j}(\pi)).$$

Here, $\Gamma_v(s)$ is defined by

$$\Gamma_v(s) = N_v(N_v \pi)^{-s} \Gamma\left(\frac{N_v s}{2}\right)$$

with $N_v = 1$ if $v \in S_f$ and $N_v = 2$ otherwise and $\mu_{v,j}(\pi)$ is a complex number determined by $\pi_v$ for each $v \in S_\infty$. The number $d_\pi = d_{L(s, \pi)} = d \sum_{v \in S_\infty} N_v$ denotes the degree of the function $L(s, \pi).$ We note that $\text{Re}(\mu_{v,j}(\pi)) > -\frac{1}{2}$. It is known that $\Lambda(s, \pi)$ can be continued analytically to the whole complex plane $\mathbb{C}$ except in the case $d_\pi = 1$, and that $\pi$ is the trivial character $\mathbf{1}$ for which $L(s, \pi)$ is the Dedekind zeta function $\zeta_K(s)$ of $K$ and $\Lambda(s, \pi)$ has simple poles at $s = 0$ and $s = 1$. Moreover, it satisfies the functional equation

$$N_\pi^{\frac{3}{2}} \Lambda(s, \pi) = e_\pi N_\pi^{\frac{1}{2}} \Lambda(1 - s, \pi),$$

where $N_\pi \geq 1$ is called the conductor of $\pi$, $e_\pi$ is the root number which is of modulus $1$ and $\pi$ is the contragredient representation of $\pi$. Since we look for uniform estimates for $n_\pi$ and the height of the lowest zero of $L(s, \pi)$ on the critical line, it turns out that the results can be expressed conveniently in terms of the analytic conductor $N_\pi[5, p. 713]$ defined by

$$N_\pi = N_\pi \prod_{v \in S_\infty} \prod_{j=1}^{d} (1 + |\mu_{v,j}(\pi)|^{N_v}).$$

The Generalized Ramanujan Conjecture (GRC) asserts that if $v$ is a place where $\pi_v$ is unramified, then $|\alpha_{v,j}(\pi)| = 1$ and $\text{Re}(\mu_{v,j}(\pi)) = 0$. Unconditionally, Jacquet and Shalika [6] proved the bounds

$$q_v^{-1/2} < |\alpha_{v,j}(\pi)| < q_v^{1/2},$$

and a similar local analysis for archimedean places would give $|\text{Re}(\mu_{v,j}(\pi))| < \frac{1}{2}$. The best bound for general $GL_d$ is due to Luo, Rudnick, and Sarnak [7]. The Ramanujan bound has been proven in very few cases. For instance, the most prominent among them are holomorphic forms on $GL_2$ and $GSp_4$. See [2] for a survey of what progress is known towards proving the Ramanujan bound.

2. The Weil explicit formula

The Weil explicit formula for an $L$-function is a tool that gives a relation between a function evaluated at the zeros of an $L$-function and the Fourier transform of that function evaluated at logarithms of prime powers, with some additional structure related to the global nature of the $L$-function. By following the strategy of Iwaniec and Kowalski [4, Section 5.5], we can formulate the following form of the explicit formula. For $T > 0$, let $\mathcal{R}(\pi)$ be the set of non-trivial zeros of $L(s, \pi)$.

**Lemma 1.** Let $Q > 1$ and $\phi(x)$ be a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ whose Fourier transform $\hat{\phi}(y) = \int_{-\infty}^{\infty} \phi(x)e^{-2\pi i xy} dx$ has compact support (in particular, $\phi$ can be extended as a smooth function on $\mathbb{C}$). Then, it holds that
Then, where

\[ H_{v,j}(Q, \phi, \pi) = \int_{-\infty}^{\infty} \phi(t) \left( \frac{\Gamma'_v}{\Gamma_v} \left( \frac{1}{2} + \mu_{v,j}(\pi) \right) + \frac{2\pi i t}{\log Q} \right) \left( \frac{1}{2} + \mu_{v,j}(\pi) - \frac{2\pi i t}{\log Q} \right) dt \]

and \( \delta_{1,1} = \delta_{1,1}(\pi) = 1 \) if \( d_{\pi} = 1 \) or \( \pi = 1 \) and 0 otherwise.

Using the same argument as Barner [1], we deduce from Lemma 1 a similar form of the Weil-type explicit formula. For a function \( F : \mathbb{R} \to \mathbb{C} \) of bounded variation (i.e., \( V_{\mathbb{R}}(F) < \infty \) where \( V_{\mathbb{R}}(F) \) is the total variation of \( F \) on \( \mathbb{R} \)), we define the function \( \Phi_F(s) \) for \( s \in \mathbb{C} \) by:

\[ \Phi_F(s) = \hat{F} \left( -\frac{s - \frac{1}{2}}{2\pi i} \right) = \int_{-\infty}^{\infty} F(x) e^{(s - \frac{1}{2})x} dx. \]

Moreover, for \( v \in S_\infty \) and \( 1 \leq j \leq d \), let \( F_{v,j}(x, \pi) = F(x)e^{-2\pi i \eta_{v,j}(\pi)x} \), \( \tilde{F}_{v,j}(x, \pi) := F_{v,j}(x, \pi) + F_{v,j}(-x, \pi) \) and \( \mu_{v,j}(\pi) = \xi_{v,j}(\pi) + i\eta_{v,j}(\pi) \) with \( \xi_{v,j}(\pi), \eta_{v,j}(\pi) \in \mathbb{R} \).

Theorem 2.1. Let \( F : \mathbb{R} \to \mathbb{C} \) be a function of bounded variation that satisfies the following conditions:

(a) there is a positive constant \( b \) such that \( V_{\mathbb{R}}(F(x)e^{(\frac{1}{2} + b|x|)}) < \infty \);

(b) \( F \) is “normalized”, that is, \( 2F(x) = F(x + 0) + F(x - 0) \) for \( x \in \mathbb{R} \);

(c) for any \( v \in S_\infty \) and \( 1 \leq j \leq d \), \( \tilde{F}_{v,j}(x, \pi) = 2F(0) + O(|x|) \) as \( |x| \to 0 \).

Then, we have

\[
\begin{align*}
\sum_{\rho \in R(\pi)} \Phi_F(\rho) &= F(0) \log \frac{N_{\pi}}{(2^{\log_{2} N_{\pi}})^d} + (\Phi_F(0) + \Phi_F(1))\delta_{1,1} + \sum_{v \in S_\infty} \sum_{j=1}^{d} W_{v,j}(F, \pi) \\
&- \sum_{v \in S_f} \sum_{l=1}^{\infty} \left( \frac{A_{\pi}(q_{v,l})}{q_{v,l}^2} F(l \log q_{v,l}) + \frac{A_{\pi}(q_{v,l})}{q_{v,l}^2} F(-l \log q_{v,l}) \right),
\end{align*}
\]

(1)

where

\[
W_{v,j}(F, \pi) = \int_{0}^{\infty} \left( \frac{N_{v}F(0)}{x} - \tilde{F}_{v,j}(x, \pi) e^{\frac{\pi}{\pi} - \frac{1}{2} - \xi_{v,j}(\pi) x} 1 - e^{-\frac{\pi}{\pi} x} \right) e^{-\frac{\pi}{\pi} x} dx.
\]

Proof. Replace \( Q = e^{2\pi} \) and \( \phi(x) = \hat{F} \left( -\frac{x}{2\pi} \right) \) in Lemma 1 and using that \( \hat{F}(y) = 2\pi F(2\pi y) \), we obtain

\[
\begin{align*}
\sum_{\rho \in R(\pi)} \Phi_F(\rho) &= F(0) \log N_{\pi} + (\Phi_F(0) + \Phi_F(1))\delta_{1,1} + \sum_{v \in S_\infty} \sum_{j=1}^{d} Y_{v,j}(F, \pi) \\
&- \sum_{v \in S_f} \sum_{l=1}^{\infty} \left( \frac{A_{\pi}(q_{v,l})}{q_{v,l}^2} F(l \log q_{v,l}) + \frac{A_{\pi}(q_{v,l})}{q_{v,l}^2} F(-l \log q_{v,l}) \right),
\end{align*}
\]

where

\[
Y_{v,j}(F, \pi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F} \left( -\frac{t}{2\pi} \right) \left( \frac{\Gamma'_v}{\Gamma_v} \left( \frac{1}{2} + \mu_{v,j}(\pi) + it \right) + \frac{\Gamma'_v}{\Gamma_v} \left( \frac{1}{2} + \mu_{v,j}(\pi) - it \right) \right) dt.
\]
Notice that both conditions (a) and (b) guarantee the convergence of the infinite sum $\sum_{\rho \in \mathbb{R}(s)} \Phi_\tau(\rho)$ (more precisely, see [1]). Now, we compute the integral $Y_{\nu,j}(F, \pi)$. Since $\mu_{\nu,j}(\pi) = \overline{\mu}_{\nu,j}(\pi) = \xi_{\nu,j}(\pi) - i\eta_{\nu,j}(\pi)$ and using the formula $\Gamma'(s) = -N_s/2\log N_s + N_s/2 \Gamma(N_s/2)$, we have:

$$Y_{\nu,j}(F, \pi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[ \tilde{\Gamma}(\frac{t}{2\pi}) + \tilde{\Gamma}(\frac{t}{2\pi}) \right] \left( \frac{1}{2} + \xi_{\nu,j}(\pi) + it \right) dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Gamma}_{\nu,j}(\pi) \left( \frac{1}{2} + \xi_{\nu,j}(\pi) + it \right) dt$$

$$= F(0) \log \frac{1}{(N_s\pi)^{N_s}} + \frac{N_s}{2\pi} \int_{-\infty}^{\infty} \hat{\Gamma}_{\nu,j}(\pi) \left( \frac{1}{2} + \xi_{\nu,j}(\pi) + it \right) dt. \quad (2)$$

Here, for $a, b > 0$ and $G \in L^1(\mathbb{R})$ satisfying $V_\mathbb{R}(G) < \infty$ and $G(x) = G(0) + O(|x|)$ as $s \to 0$, the following formula was also established in [1]:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Gamma}(\frac{t}{2\pi}) \left( a + \frac{t}{b} \right) dt = \int_{0}^{\infty} \frac{G(0)}{x} \hat{\Gamma}(\pi - \frac{t}{x}) \left( a + \frac{t}{b} \right) e^{-bx} dx.$$

Using the assumption (c) of Theorem 2.1, we can apply the above formula with $G = \hat{\Gamma}_{\nu,j}, a = \frac{N_s}{2} \left( \frac{1}{2} + \xi_{\nu,j}(\pi) \right)$ and $b = \frac{2}{N_s}$ and obtain:

$$Y_{\nu,j}(F, \pi) = F(0) \log \frac{1}{(N_s\pi)^{N_s}} + W_{\nu,j}(F, \pi).$$

This completes the proof. We may also point out that similar explicit formulas were established by Mestre [8] for rather general $L$-functions. □

3. The lowest zero of $L$-functions

Theorem 2.1 makes it possible to prove under the Riemann hypothesis that the lowest zero of $L(s, \pi)$ tends to $1/2$ when the analytic conductor $N_\pi$ is large. To do so, we first give a conditional improvement of the upper bound for the vanishing order $n_\pi$ of $L(s, \pi)$ at $s = 1/2$. This yields an estimate for the imaginary part $\gamma_\pi$ of the lowest zero $\rho_\pi = 1/2 + i\gamma_\pi$ of $L(s, \pi)$ distinct from $1/2$. For this purpose, we apply Theorem 2.1 to suitable functions with compact support. If we assume the Riemann hypothesis, then one can prove more precise estimates on $\gamma_\pi$. Such improvements have been also considered by Mestre [8] for the elliptic curve $L$-functions, the author [9] for Dedekind zeta functions and Iwaniec and Kowalski [4, Proposition 5.21] as an application of the positivity technique in the explicit formula.

Theorem 3.1. Assuming the Riemann hypothesis, we have for large $N_\pi$:

$$n_\pi \ll \frac{\log N_\pi}{\log(\frac{3}{2} \log N_\pi)} \quad \text{and} \quad |\gamma_\pi| \ll \frac{1}{\log(\frac{3}{2} \log N_\pi)}.$$

Proof. We first need an estimate for the sum over the finite places of $K$ in (1). Let $F$ be a function of support contained in $[-1, 1]$ satisfying the hypotheses of Theorem 2.1 and let $F_T(x) = F(x/T)$, then $\tilde{F}_T(u) = T \tilde{F}(u)$. By using the classical prime number theorem one can prove the following estimate.

Lemma 2. The sum over $\nu \in S_f$ in (1) is bounded as follows:

$$\left| \sum_{\nu \in S_f} \sum_{i=1}^{\infty} \left( \frac{A_\pi(q_\nu)}{q_\nu^i} F_T(i \log q_\nu) + \frac{A_\pi(q_\nu)}{q_\nu^i} F_T(-i \log q_\nu) \right) \right| \ll T.$$  

Actually, since $|\alpha_{\nu,j}(\pi)| < q_\nu^{1/2}$, we have $|A_\pi(n)| \leq dA(n)n^{1/2}$. Therefore, using the prime number theorem, the sum over $\nu \in S_f$ in (1) is bounded by

$$2d \sum_{\log n \leq T} A(n) \ll T.$$
where the implied constant is absolute. Let \( f \) be a function defined by
\[
f(x) = \begin{cases} 
1 - |x| & \text{if } |x| < 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Then, \( f \) satisfies the hypothesis of Theorem 2.1 and
\[
\hat{f}(u) = \left( \frac{2 \sin(u/2)}{u} \right)^2.
\]

Therefore, by applying Theorem 2.1 to \( f_T \), we obtain:
\[
n^{1/2}_\pi T \leq \delta_{1,1} e^{T/2} - 2 \sum_{n \geq 1} \frac{\text{Re}(\Lambda_{\pi}(n))}{n^2} F_T(\log n) + O(\log N_\pi).
\]  
(3)

By using Lemma 2 and replacing \( T \) by \( \sqrt{2}/\gamma_\pi \) in (3), we have for large \( N_\pi \):
\[
n^{1/2}_\pi \ll \frac{\log N_\pi}{\log \left( \frac{3}{d \log N_\pi} \right)}.
\]

Then, the first assertion of Theorem 3.1 is proved. In order to prove the second assertion of Theorem 3.1, we need another even function supported on \([-1, 1]\).
\[
g(x) = \begin{cases} 
(1 - x) \cos \pi x + \frac{3}{\pi} \sin \pi x & \text{if } x \in [0, 1] \\
0 & \text{otherwise}.
\end{cases}
\]

Note that \( g \) satisfies the conditions of Theorem 2.1, then
\[
\hat{g}(u) = \left( 2 - \frac{u^2}{\pi^2} \right) \left[ \frac{2\pi}{\pi^2 - u^2} \cos \frac{u}{2} \right]^2.
\]

Applying Theorem 2.1 with \( g_T(x) = g(x/T) \) and replacing \( T \) by \( \sqrt{2}/\gamma_\pi \), we obtain:
\[
\frac{8}{\pi^2} n^{1/2}_\pi T - (\Phi_{g_T}(0) + \Phi_{g_T}(1)) \delta_{1,1} + 2 \sum_{n \geq 1} \frac{\text{Re}(\Lambda_{\pi}(n))}{n^2} g_T(\log n) \gg \log N_\pi.
\]  
(4)

Using Lemma 2, the last estimate of \( n_\pi \), we deduce from (4) the following inequality for some constants \( A \) and \( B \):
\[
\frac{\log N_\pi}{\log \left( \frac{3}{d \log N_\pi} \right)} A T + B d e^T \gg \log N_\pi.
\]

Therefore, for sufficiently large \( N_\pi \), we get
\[
T \gg \log \left( \frac{3}{d \log N_\pi} \right),
\]
so
\[
|\gamma_\pi| \ll \frac{1}{\log \left( \frac{3}{d \log N_\pi} \right)}.
\]

As a consequence, one can show that any fixed interval on the critical line around \( s = 1/2 \) contains zeros of \( L(s, \pi) \) when \( N_\pi \) is sufficiently large. \( \square \)

**Corollary 1.** Assuming the Riemann hypothesis, we have:
\[
\lim_{N_\pi \to +\infty} \rho_\pi = \frac{1}{2}.
\]

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References