Complex analysis

Weak solutions to complex Monge–Ampère equations on compact Kähler manifolds

Solutions faibles des équations de Monge–Ampère complexes sur des variétés de Kähler compactes

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ABSTRACT

We show a general existence theorem of solutions to the complex Monge–Ampère type equation on compact Kähler manifolds.

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RÉSUMÉ

Nous montrons un théorème général d’existence et d’unicité de solution d’une équation de type Monge–Ampère complexe sur des variétés de Kähler compactes.

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1. Introduction

Let \((X, \omega)\) be a compact Kähler manifold of dimension \(n\). Throughout this note, \(\theta\) denotes a smooth closed form of bidegree \((1, 1)\) which is nonnegative and big, i.e. such that \(\int_X \theta^n > 0\). Recall that a \(\theta\)-plurisubharmonic (\(\theta\)-psh for short) function is an upper semi-continuous function \(\varphi\) such that \(\theta + \ddc \varphi\) is nonnegative in the sense of currents. The set of all \(\theta\)-psh functions \(\varphi\) on \(X\) will be denoted by \(PSH(X, \theta)\) and endowed with the weak topology, which coincides with the \(L^p(X)\)-topology. We shall consider the existence and uniqueness of the weak solution to the following complex Monge–Ampère equations

\[
(\theta + \ddc \varphi)^n = F(\varphi, \cdot) d\mu
\]

(1)

where \(\varphi\) is a \(\theta\)-psh function, \(F(t, x) \geq 0\) is a measurable function on \(\mathbb{R} \times X\) and \(\mu\) is a positive measure. It is well known that we cannot always make sense to the left-hand side of (1) as a nonnegative measure. But according to [4] (see also [6,7,12]), we can define the non-pluripolar product \(\theta + \ddc u^n\) as the limit of \(1_{(u > -j)}(\theta + \ddc (\max(u, -j)))^n\). It was shown in [7] that its trivial extension is a nonnegative closed current and

\[
\int_X (\theta + \ddc u)^n \leq \int_X \theta^n.
\]

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Denote by $\mathcal{E}(X, \theta)$ the set of all $\theta$-psh with full non-pluripolar Monge–Ampère measure, i.e. the $\theta$-psh functions for which the last inequality becomes an equality.

For $F$ smooth and $\mu = dV$ is a smooth positive volume form, the equation has been studied extensively by various authors, see for example [1,2,7,15,13,14,16], etc., and references therein. Recently, Kołodziej treated the case $F$ bounded by a function independent of the first variable and $\mu = e^{\omega}$, where $\omega$ is a Kähler form on $X$. In this paper, we consider a more general case. Our main purpose is to prove the following theorem.

**Main Theorem.** Assume that $F : \mathbb{R} \times X \to [0, +\infty)$ is a measurable function such that:

1) for all $x \in X$ the function $t \mapsto F(t, x)$ is continuous and nondecreasing;
2) $F(t, \cdot) \in L^1(X, d\mu)$ for all $t \in \mathbb{R}$;
3) $\lim_{t \to -\infty} \int_X F(t, x) \, d\mu \leq \int_X \theta^n < \lim_{t \to +\infty} \int_X F(t, x) \, d\mu$.

Then there exists a unique (up to additive constant) $\theta$-psh function $\varphi \in \mathcal{E}(X, \theta)$ solution to

$$ (\theta + dd^c \varphi)^n = F(\varphi, \cdot) \, d\mu. $$

**2. Proof**

**Lemma 2.1.** Let $\mu$ be a positive measure on $X$ vanishing on all pluripolar subsets of $X$ and $u_j \in \mathcal{E}(X, \theta)$ such that $u_j \geq u_0$ for some $u_0 \in \mathcal{E}(X, \theta) \cap L^1(d\mu)$. If $u_j \to u$ in $L^1(X)$, then

$$ \lim_{j \to +\infty} \int_X u_j \, d\mu = \int_X u \, d\mu. $$

**Proof.** Since $u_0 \in L^1(d\mu)$ and the measure $\mu$ puts no mass on pluripolar subsets of $X$, then

$$ \int_{(u_j \leq -\alpha)}^\infty \int_{(u_j \leq -\alpha)} \, d\mu \, dt \leq \int_{(u_0 \leq -\alpha)}^\infty \int_{(u_0 \leq -\alpha)} \, d\mu \, dt \to 0 \quad \text{as } \alpha \to +\infty. $$

Hence, by the Dunford–Petit theorem (see for example [10] p. 274), we have that the sequence $(u_j)$ is weakly relatively compact in $L^1(d\mu)$. Let $\hat{u} \in L^1(d\mu)$ be a cluster point of $(u_j)$. After extracting a subsequence, we may assume that $(u_j)$ converges to $\hat{u}$ weakly in $L^1(d\mu)$. On the other hand, we have $u_j \to u$ in $L^1(X)$. So, choosing a subsequence if necessary, we can assume that $u_j \to u$ point-wise on $X \setminus A$, where $A = \{ \limsup_{j \to \infty} u_j < u \}$. But $A$ is negligible, hence, by [3] $A$ is pluripolar subset of $X$, thus $\mu(A) = 0$. It follows from Lebesgue’s dominated convergence theorem that $u_j \to u$ weakly in $L^1(d\mu)$. Therefore $\hat{u} = uu_\mu$-a.e. Hence $u$ is the unique cluster point of $(u_j)$, which means that $(u_j)$ converges to $u$ weakly in $L^1(d\mu)$ and the proof is complete. $\square$

The following corollary is the global version of Corollary 1.4 in [8].

**Corollary 2.2.** Let $\mu$ be a nonnegative measure that puts no mass on pluripolar sets of $X$. Then for any sequence $u_j \in \mathcal{E}(X, \theta)$ converging weakly, one can extract a subsequence that converges pointwise $d\mu$-almost everywhere.

**Proof of the Main Theorem.** The set of $\varphi \in PSH(X, \theta)$ normalized by $\sup_X \varphi = 0$ is compact (cf. [11,12]). Then there exists a positive constant $C_0 > 0$ such that

$$ \int_X -u \theta^n \leq C_0, \quad \forall u \in PSH(X, \theta); \quad \sup_X u = 0. $$

Consider the set

$$ \mathcal{H} := \left\{ \varphi \in PSH(X, \theta); \varphi \leq 0 \text{ and } \int_X -\varphi \theta^n \leq C_0 \right\} $$

It is obvious that $\mathcal{H}$ is a compact convex subset of $L^1(X)$. 
From the conditions of the main theorem, there exists a real number \( c_0 \) such that
\[
\int_X F(c_0, \cdot) d\mu = \int_X \theta^n.
\]

Fix a function \( \phi \in \mathcal{H} \). Then there exists a real number \( c_\phi \geq c_0 \) such that
\[
\int_X F(\phi + c_\phi, \cdot) d\mu = \int_X \theta^n.
\]

Since \( F(\phi + c_\phi, \cdot) \in L^1(X, d\mu) \) and \( \mu \) vanishes on pluripolar sets, it follows by [7,5] that there exists a function \( \tilde{\phi} \in \mathcal{E}(X, \theta) \) such that \( \sup_X \tilde{\phi} = 0 \) and
\[
(\theta + dd^c \tilde{\phi})^n = F(\phi + c_\phi, \cdot) d\mu.
\]

The function \( \tilde{\phi} \) does not depend on the constant \( c_\phi \). Indeed, assume that there exist two constant \( c_\phi \) and \( c'_\phi \) such that
\[
\int_X F(\phi + c_\phi, \cdot) d\mu = \int_X F(\phi + c'_\phi, \cdot) d\mu = \int_X \theta^n.
\]

If \( c_\phi \leq c'_\phi \) then \( F(\phi + c_\phi, \cdot) d\mu \leq F(\phi + c'_\phi, \cdot) d\mu \). Thence
\[
F(\phi + c_\phi, \cdot) d\mu = F(\phi + c'_\phi, \cdot) d\mu.
\]

By the uniqueness result in [7] and [9], we get that \( \tilde{\phi} \) is unique and therefore independent of the constant \( c_\phi \).

From the definition of \( \mathcal{H} \) we have \( \phi \in \mathcal{H} \). Consider the map \( \Phi : \mathcal{H} \to \mathcal{H} \) defined by \( \phi \mapsto \tilde{\phi} \). In fact, the range of \( \Phi \) is equal to \( \mathcal{H} \cap \mathcal{E}(X, \theta) \).

We claim that \( \Phi \) is continuous. Indeed, let \( \psi_j \in \mathcal{H} \) be a converging sequence with limit \( \psi \in \mathcal{H} \) in \( L^1(X) \)-topology. Let \( \psi \) be any cluster point of the sequence \( \tilde{\psi}_j := \Phi(\psi_j) \). We may assume, up to extracting, that \( \tilde{\psi}_j \) converges towards \( \psi \) in \( L^1(X) \). Since the measure \( \mu \) vanishes on pluripolar subsets, then by Corollary 2.2 above, we can extract a subsequence, which is still denoted by \( \psi_j \), so that \( \psi_j \to \psi \) \( \mu \)-a.e. We claim that the sequence \( c_{\psi_j} \) is bounded. Indeed, by construction we have \( c_{\psi_j} \geq c_0 \). Now if \( c_{\psi_j} \to +\infty \) then
\[
\int_X \theta^n = \liminf_{j \to +\infty} \int_X F(\psi_j + c_{\psi_j}, \cdot) d\mu > \int_X \theta^n,
\]

which is impossible.

So by passing to a subsequence, we may assume that \( c_{\psi_j} \to c_0 \). Therefore \( F(\psi_j + c_{\psi_j}, \cdot) \to F(\psi + c, \cdot) \) in \( L^1(d\mu) \). Since \( \tilde{\psi}_j \to \psi \) in \( L^1(X) \), then \( \psi = (\limsup_{j \to +\infty} \tilde{\psi}_j)^\ast \) and therefore by Hartogs’ lemma \( \sup_X \psi = 0 \). Let denote \( \psi_j := (\sup_{k \geq j} \tilde{\psi}_k)^\ast = (\limsup_{k \to +\infty} \max_{l \geq k} \tilde{\psi}_k)^\ast \). Since the set \( \{ \sup_{k \geq j} \tilde{\psi}_k < (\sup_{k \geq j} \tilde{\psi}_k)^\ast \} \) is pluripolar, then by the continuity of the complex Monge–Ampère operator along monotonic sequences, we have:
\[
(\theta + dd^c \psi_j)^n = \lim_{j \to +\infty} (\theta + dd^c \tilde{\psi}_j)^n
\]
\[
= \lim_{j \to +\infty} \lim_{l \to +\infty} (\theta + dd^c \max_{l \geq k \geq j} \tilde{\psi}_k)^n
\]
\[
\geq \lim_{j \to +\infty} \liminf_{l \to +\infty} F(\psi_j + c_{\psi_j}, \cdot) d\mu
\]
\[
= \liminf_{j \to +\infty} F(\psi_j + c_{\psi_j}, \cdot) d\mu
\]
\[
= F(\psi + c, \cdot) d\mu.
\]

Thence \( (\theta + dd^c \psi)^n = (\theta + dd^c \tilde{\psi})^n \). By uniqueness (shown in [7]), we get \( \Phi = \psi \) and therefore \( \Phi \) is continuous. Now, Shauder’s fixed point theorem implies that there exists a function \( u \in \mathcal{H} \) such that \( \Phi(u) = u \). Since \( \Phi(\mathcal{H}) \subset \mathcal{E}(X, \theta) \) we have \( u \in \mathcal{E}(X, \theta) \) and
\[
(\theta + dd^c u)^n = F(u + c_u, \cdot) d\mu.
\]

The function \( \varphi := u + c_u \) is the required solution.

Uniqueness follows in a classical way from the comparison principle [3] and its generalization [7]. Indeed assume that there exist two solutions \( \varphi_1 \) and \( \varphi_2 \) in \( \mathcal{E}(X, \theta) \) such that
\[
(\theta + dd^c \varphi_i)^n = F(\varphi_i, \cdot), \quad i = 1, 2.
\]
Then, since $F$ is non-decreasing with respect to the first variable, we have
\[
F(\varphi_1, \cdot) \, d\mu \leq F(\varphi_2, \cdot) \, d\mu \quad \text{on } (\varphi_1 < \varphi_2).
\] (3)

On the other hand, by the comparison principle we have
\[
\int_{(\varphi_1 < \varphi_2)} \left( \theta + dd^c \varphi_2 \right)^n \leq \int_{(\varphi_1 < \varphi_2)} \left( \theta + dd^c \varphi_1 \right)^n.
\] (4)

Combining (2), (3) and (4), we get
\[
\int_{(\varphi_1 < \varphi_2)} F(\varphi_1, \cdot) \, d\mu \leq \int_{(\varphi_1 < \varphi_2)} F(\varphi_2, \cdot) \, d\mu \leq \int_{(\varphi_1 < \varphi_2)} F(\varphi_1, \cdot) \, d\mu.
\]

Hence
\[
F(\varphi_1, \cdot) = F(\varphi_2, \cdot) \mu\text{-almost everywhere} \quad \text{on } (\varphi_1 < \varphi_2).
\]

In the same way, we get the equality on $(\varphi_1 > \varphi_2)$ and then on $X$. Hence
\[
\left( \theta + dd^c \varphi_1 \right)^n = \left( \theta + dd^c \varphi_2 \right)^n.
\]

Once more, the uniqueness result of [7] and [9] implies that $\varphi_1 - \varphi_2 = \text{Cst.}$ \hfill $\square$

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