Partial differential equations

Non-existence of local solutions of semilinear heat equations of Osgood type in bounded domains

Non-existence de solutions locales pour les équations de la chaleur semi-linéaires de type Osgood dans des domaines bornés

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**A B S T R A C T**

We establish a local non-existence result for the equation \( u_t - \Delta u = f(u) \) with Dirichlet boundary conditions on a smooth bounded domain \( \Omega \subset \mathbb{R}^n \) and initial data in \( L^q(\Omega) \) when the source term \( f \) is non-decreasing and \( \lim_{s \to -\infty} s^{-\gamma} f(s) = \infty \) for some exponent \( \gamma > q(1 + 2/n) \). This allows us to construct a locally Lipschitz \( f \) satisfying the Osgood condition \( \int_1^\infty 1/f(s) \, ds = \infty \), which ensures global existence for initial data in \( L^\infty(\Omega) \), such that for every \( q \) with \( 1 \leq q < \infty \) there is a non-negative initial condition \( u_0 \in L^q(\Omega) \) for which the corresponding semilinear problem has no local-in-time solution ('immediate blow-up').

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**R É S U M É**

Nous établissons un résultat de non-existence locale pour l'équation \( u_t - \Delta u = f(u) \) avec des conditions aux limites de Dirichlet sur un domaine borné lisse \( \Omega \subset \mathbb{R}^n \) et des données initiales dans \( L^q(\Omega) \) lorsque le terme de source \( f \) est non décroissant et \( \lim_{s \to -\infty} s^{-\gamma} f(s) = \infty \) pour un exposant \( \gamma > q(1 + 2/n) \). Ceci nous permet de construire un \( f \) localement Lipschitz qui satisfait la condition d'Osgood \( \int_1^\infty 1/f(s) \, ds = \infty \), ce qui garantit l'existence globale pour des données initiales dans \( L^\infty(\Omega) \), de telle sorte que pour chaque \( q \) tel que \( 1 \leq q < \infty \) il existe une condition initiale non négative \( u_0 \in L^q(\Omega) \) pour laquelle le problème semi-linéaire correspondant n'admet pas de solution locale en temps ('blow-up immédiat').

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1. Introduction

In the previous paper [8] we showed that for locally Lipschitz $f$ with $f > 0$ on $(0, \infty)$, the Osgood condition
\[ \int_1^\infty \frac{1}{f(s)} \, ds = \infty, \] (1)
which ensures global existence of solutions of the scalar ODE $\dot{x} = f(x)$, is not sufficient to guarantee the local existence of solutions of the 'toy PDE'
\[ u_t = f(u), \quad u(x, 0) = u_0 \in L^q(\Omega) \] (2)
unless $q = \infty$.

In [4] we considered the Cauchy problem for the semilinear PDE
\[ u_t = \Delta u + f(u), \quad u(0) = u_0. \] (3)
on the whole space $\mathbb{R}^n$ and showed that even with the addition of the Laplacian, for each $q$ with $1 \leq q < \infty$ one can find a non-negative, locally Lipschitz $f$ satisfying the Osgood condition (1) such that there are non-negative initial data in $L^q(\mathbb{R}^n)$ for which there is no local-in-time integrable solution of (3).

In this paper we obtain a similar non-existence result for equation (3) when posed with Dirichlet boundary conditions on a smooth bounded domain $\Omega \subset \mathbb{R}^n$. More explicitly, we focus throughout the paper on the following problem:
\[ u_t = \Delta u + f(u), \quad u|_{\partial\Omega} = 0, \quad u(x, 0) = u_0 \in L^q(\Omega). \] (P)
In all that follows we assume that the source term $f : [0, \infty) \to [0, \infty)$ is non-decreasing. We show in Theorem 3.2 that if $f$ satisfies the asymptotic growth condition
\[ \lim_{s \to \infty} s^{-\gamma} f(s) = \infty \] (4)
for some $\gamma > q(1 + 2/n)$ then one can find a non-negative $u_0 \in L^q(\Omega)$ such that there is no local-in-time solution of (P). We then (Theorem 4.1) construct a Lipschitz function $f$ that grows quickly enough such that (4) holds for every $\gamma \geq 0$, but nevertheless still satisfies the Osgood condition (1). This example shows that there are functions $f$ for which (P) has solutions for any $u_0$ belonging to $L^\infty(\Omega)$, but that there are non-negative $u_0 \in L^q(\Omega)$ for any $1 \leq q < \infty$ for which the equation has no local integral solution.

One can see this result as in some sense dual to that of Fila et al. [3] (see also Section 19.3 of [7]), who show that there exists an $f$ such that all positive solutions of $\dot{x} = f(x)$ blow up in finite time while all solutions of (P) are global and belong to $L^\infty(\Omega)$.

2. A lower bound on solutions of the heat equation

Without loss of generality we henceforth assume that $\Omega$ contains the origin. For $r > 0$, $B(r)$ will denote the Euclidean ball in $\mathbb{R}^n$ of radius $r$ centred at the origin, and $\omega_n$ the volume of the unit ball in $\mathbb{R}^n$.

As an ingredient in the proof of Theorem 3.2, we want to show that the action of the heat semigroup on the characteristic function of a ball
\[ \chi_{B}(x) = \begin{cases} 
1 & \text{for } x \in B(R) \\
0 & \text{for } x \notin B(R) 
\end{cases} \]
does not have too pronounced an effect for short times.

We denote the solution of the heat equation on $\Omega$ at time $t$ with initial data $u_0$ by $S_{\Omega}(t)u_0$, i.e. the solution of
\[ u_t - \Delta u = 0, \quad u|_{\partial\Omega} = 0, \quad u(x, 0) = u_0 \in L^q(\Omega). \]
This solution can be given in terms of the Dirichlet heat kernel $K_\Omega(x, y; t)$ by the integral expression
\[ [S_{\Omega}(t)u_0](x) = \int_{\Omega} K_\Omega(x, y; t)u_0(y) \, dy. \]
We note for later use that $K_\Omega(x, y; t) = K_\Omega(y, x; t)$ for all $x, y \in \Omega$.

We use the following Gaussian lower bound on the Dirichlet heat kernel, which is obtained by combining various estimates proved by van den Berg in [9] (Theorem 2 and Lemmas 8 and 9). A simplified proof is given in [5].
Lemma 2.1. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \), and denote by \( K_\Omega(x, y; t) \) the Dirichlet heat kernel on \( \Omega \). Suppose that
\[
\epsilon := \inf_{z \in \{x, y\}} \text{dist}(z, \partial \Omega) > 0,
\]
where \( [x, y] \) denotes the line segment joining \( x \) and \( y \) (so in particular \( [x, y] \) is contained in the interior of \( \Omega \)). Then for \( 0 < t \leq \epsilon^2/n^2 \)
\[
K_\Omega(x, y; t) \geq \frac{1}{4} \frac{1}{G_n(x, y; t)}, \quad \text{where} \quad G_n(x, y; t) = \left(4\pi t\right)^{-n/2} e^{-\|x-y\|^2/4t}.
\]

We can now bound \( S_\Omega(t)\chi_R \) from below.

Lemma 2.2. There exists an absolute constant \( c_n > 1 \), which depends only on \( n \), such that for any \( R \) for which \( B(2R) \subset \Omega \),
\[
S_\Omega(t)\chi_R \geq \frac{1}{c_n} \chi_{K/2}, \quad \text{for all} \quad 0 < t \leq R^2/n^2.
\]

Proof. Take \( x \in B(R/2) \); then when \( y \in B(R) \) certainly \( \epsilon \geq R \), so \( (6) \) implies that for \( 0 < t \leq R^2/n^2 \)
\[
\left[ S_\Omega(t)\chi_R \right](x) = \int_{B(R)} K_\Omega(x, y; t) \, dy \geq \frac{1}{4} \left(4\pi t\right)^{-n/2} \int_{B(R)} e^{-\|x-y\|^2/4t} \, dy.
\]
Since \( |x| \leq R/2 \), it follows that \( \|w = x - y : y \in B(R)\| \supset B(R/2) \) and so
\[
\left[ S_\Omega(t)\chi_R \right](x) \geq \frac{1}{4} \pi^{-n/2} \int_{B(R/2)} e^{-|w|^2/4t} \, dw = \frac{1}{4} \pi^{-n/2} \int_{B(R/4)} e^{-|z|^2} \, dz =: c_n^{-1},
\]

3. Non-existence of local solutions

In this section we prove the non-existence of local \( L^q \)-valued solutions, taking the following definition from [7] as our (essentially minimal) definition of such a solution. Note that any classical or mild solution is a local integral solution in the sense of this definition [7, pp. 77–78], and so non-existence of a local \( L^q \)-valued integral solution implies the non-existence of classical and mild \( L^q \)-valued solutions.

Definition 3.1. Given \( f \geq 0 \) and \( u_0 \geq 0 \) we say that \( u \) is a local integral solution of (P) on \([0, T)\) if \( u : \Omega \times [0, T) \rightarrow [0, \infty) \) is measurable, finite almost everywhere, and satisfies
\[
u(t) = S_\Omega(t)u_0 + \int_0^t S_\Omega(t-s) f(u(s)) \, ds
\]
almost everywhere in \( \Omega \times [0, T) \). We say that \( u \) is a local \( L^q \)-valued integral solution if in addition \( u(t) \in L^q(\Omega) \) for almost every \( t \in (0, T) \).

We now prove our main result, in which we obtain non-existence of a local \( L^q \)-valued integral solution for certain initial data in \( L^q(\Omega) \), \( 1 \leq q < \infty \), under the asymptotic growth condition (9) when \( f \) is non-decreasing.

Theorem 3.2. Let \( q \in (1, \infty) \). Suppose that \( f : [0, \infty) \rightarrow [0, \infty) \) is non-decreasing. If
\[
\limsup_{s \to \infty} s^{-\gamma} f(s) = \infty
\]
for some \( \gamma > q(1 + \frac{2}{n}) \), then there exists a non-negative \( u_0 \in L^q(\Omega) \) such that (P) possesses no local \( L^q \)-valued integral solution.

Proof. We find a \( u_0 \in L^q(\Omega) \) such that \( u(t) \notin L^1_{\text{loc}}(\Omega) \) for all sufficiently small \( t > 0 \) and hence \( u(t) \notin L^q(\Omega) \) for all sufficiently small \( t > 0 \). Note that this is a stronger form of ill-posedness than ‘norm inflation’ (cf. Bourgain & Pavlović [1]).
Set \( \alpha = (n+2)/\gamma < n/q \), so that
\[
\limsup_{s \to \infty} s^{-(n+2)/\alpha} f(s) = \infty.
\]

Then in particular there exists a sequence \( \phi_i \to \infty \) such that
\[
f(\phi_i)^{-(n+2)/\alpha} \to \infty \quad \text{as} \quad i \to \infty.
\] (10)

Now choose \( R > 0 \) such that \( B(2R) \subset \Omega \) (recall that we assumed that \( 0 \in \Omega \)), and take \( u_0 = |x|^{-\alpha} \chi_R(x) \in L^q(\Omega) \). Noting that by comparison \( u(t) \geq S_\Omega(t)u_0 \geq 0 \), it follows from (8) that for every \( t > 0 \)
\[
\int_{B(R)} u(t) \, dx \geq \int_{B(R)} \int_0^t [S_\Omega(t-s) f(S_\Omega(s)u_0)](x) \, ds \, dx.
\]

Now choose and fix \( t \in (0, R^2/n^2) \). Observe that
\[
u_0 \geq \psi \chi_{\psi^{-1/\alpha}}
\]
for any \( \psi > R^{-\alpha} \). In particular, choosing \( \psi = c_n \phi_i \), it follows from Lemma 2.2 and the monotonicity of \( S_\Omega \) that for all \( i \) sufficiently large
\[
S_\Omega(s) u_0 \geq \phi_i \chi_{\frac{1}{2}(c_n \phi_i)^{-1/\alpha}}, \quad 0 \leq s \leq t_i := (c_n \phi_i)^{-2/\alpha}/n^2.
\]

Therefore, for any \( i \) large enough that \( t_i \leq t \) and \( c_n \phi_i > R^{-\alpha} \),
\[
\int_{B(R)} u(t) \, dx \geq \int_{B(R)} \int_0^{t_i} S_\Omega(t-s) f(\phi_i \chi_{\frac{1}{2}(c_n \phi_i)^{-1/\alpha}}) \, ds \, dx
\]
\[
\geq f(\phi_i) \int_{B(R)} \int_0^{t_i} S_\Omega(t-s) \chi_{\frac{1}{2}(c_n \phi_i)^{-1/\alpha}} \, dx \, ds,
\]

using Fubini’s Theorem and the fact that \( f(0) \geq 0 \).

Now observe that since \( K_\Omega(x, y; t) = K_\Omega(y, x; t) \), for any \( t > 0 \) and \( R \) such that \( B(R), B(r) \subset \Omega \),
\[
\int_{B(R)} [S_\Omega(t) \chi_r](x) \, dx = \int_{B(R)} \int_{B(r)} K_\Omega(x, y; t) \, dy \, dx = \int_{B(r)} [S_\Omega(t) \chi_r](y) \, dy.
\]

Thus
\[
\int_{B(R)} u(t) \, dx \geq f(\phi_i) \int_0^{t_i} \int_{B(\frac{1}{2}(c_n \phi_i)^{-1/\alpha})} S_\Omega(t-s) \chi_R \, dx \, ds.
\]

Since \( \frac{1}{2}(c_n \phi_i)^{-1/\alpha} < R/2 \) and \( t-s \leq t \leq R^2/n^2 \) we can use Lemma 2.2 once more to obtain
\[
\int_{B(R)} u(t) \, dx \geq f(\phi_i) \int_0^{t_i} \int_{B(\frac{1}{2}(c_n \phi_i)^{-1/\alpha})} \frac{1}{c_n} \chi_{R/2} \, dx \, ds
\]
\[
= \frac{\omega_n}{c_n} f(\phi_i) t_i \left[ \frac{1}{2} (c_n \phi_i)^{-1/\alpha} \right]^n
\]
\[
= \left[ \omega_n 2^{-n} c_n^{1-\frac{(n+2)/\alpha}{/n^2}} f(\phi_i) \phi_i^{-(n+2)/\alpha} \right] \to \infty \quad \text{as} \quad i \to \infty
\]
due to (10). \( \square \)

We note that if \( f(s) \geq cs \) for some \( c > 0 \) then arguing as in [4, Theorem 4.1] there can in fact be no local integral solution of (P) whatsoever.
For the canonical Fujita equation
\[ u_t = \Delta u + u^p, \tag{11} \]
our argument shows the non-existence of local solutions when \( p > q(1 + \frac{2}{n}) \). The sharp result in this case is known to be \( p > 1 + \frac{2q}{n} \) [11,12] with equality allowed if \( q = 1 \) [2].

The existence of a finite limit in (9) implies that \( f(s) \leq c(1 + s^r) \), and hence by comparison with (11) is sufficient for the local existence of solutions provided that \( \gamma < 1 + \frac{2q}{n} \) [10]. We currently, therefore, have an indeterminate range of \( \gamma \),
\[ 1 + \frac{2q}{n} \leq \gamma \leq q \left( 1 + \frac{2}{n} \right) \]
for which we do not know whether (9) characterises the existence or non-existence of local solutions.

4. A very 'bad' Osgood \( f \)

To finish, using a variant of the construction in [4], we provide an example of an \( f \) that satisfies the Osgood condition (1) but for which
\[ \limsup_{s \to \infty} s^{-\gamma} f(s) = \infty, \quad \text{for every } \gamma \geq 0. \tag{12} \]

**Theorem 4.1.** There exists a locally Lipschitz function \( f : [0, \infty) \to [0, \infty) \) such that \( f(0) = 0 \), \( f \) is non-decreasing, and \( f \) satisfies the Osgood condition
\[ \int_1^\infty \frac{1}{f(s)} \, ds = \infty, \]
but nevertheless (12) holds. Consequently, for this \( f \), for any \( 1 \leq q < \infty \) there exists a non-negative \( u_0 \in L^q(\Omega) \) such that (P) has no local \( L^q \)-valued integral solution.

**Proof.** Fix \( \phi_0 = 1 \) and define inductively the sequence \( \phi_i \) via
\[ \phi_{i+1} = e^{\phi_i}. \]
Clearly, \( \phi_i \to \infty \) as \( i \to \infty \). Now define \( f : [0, \infty) \to [0, \infty) \) by
\[ f(s) = \begin{cases} (e-1)s, & s \in J_0 := [0, 1], \\ \phi_i - \phi_{i-1}, & s \in J_i := [\phi_{i-1}, \phi_i/2], i \geq 1, \\ \ell_i(s), & s \in J_i := (\phi_i/2, \phi_i), i \geq 1, \end{cases} \tag{13} \]
where \( \ell_i \) interpolates linearly between the values of \( f \) at \( \phi_i/2 \) and \( \phi_i \). By construction \( f(0) = 0 \), \( f \) is Lipschitz and non-decreasing, and \( f \) is Osgood since
\[ \int_1^\infty \frac{1}{f(s)} \, ds \geq \sum_{i=1}^\infty \int_{J_i} \frac{1}{f(s)} \, ds = \sum_{i=1}^\infty \frac{\phi_i/2 - \phi_{i-1}}{\phi_i - \phi_{i-1}} = +\infty. \]

However, \( f(\phi_i) = e^{\phi_i} - \phi_i \), and so for any \( \gamma \geq 0 \)
\[ \lim_{i \to \infty} \phi_i^{-\gamma} f(\phi_i) \to \infty \text{ as } i \to \infty, \]
which shows that (12) holds. \( \square \)

This example shows that there exist semilinear heat equations that are globally well-posed in \( L^\infty(\Omega) \), yet ill-posed in every \( L^q(\Omega) \) for \( 1 \leq q < \infty \).

5. Note added in proof

Since this paper was completed we have shown that condition (9) with \( \gamma \geq 1 + \frac{2q}{n} \) is enough to find a non-negative \( u_0 \in L^q(\Omega) \) for which there exists no local solution that remains bounded in \( L^q(\Omega) \) for \( t > 0 \) [6].
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