Algebra/Group theory

A note on a characterization of generalized quaternion 2-groups

Caractérisation des 2-groupes de quaternions généralisés

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\textbf{A R T I C L E I N F O}

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\textbf{A B S T R A C T}

In this note, we answer an open problem posed in M. Tărnăceanu (2010) \cite{5}, and obtain that the generalized quaternion 2-groups are the unique finite noncyclic groups whose posets of conjugacy classes of cyclic subgroups have breaking points.

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\textbf{R É S U M É}

Répondant à une question de M. Tărnăceanu (2010) \cite{5}, nous montrons que les 2-groupes de quaternions généralisés sont les seuls groupes finis non cycliques dont le treillis des classes de conjugaison de sous-groupes cycliques admet un point clivant.

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1. Introduction

Let $G$ be a finite group and $L(G)$ be the subgroup lattice of $G$. A proper nontrivial subgroup $H$ of $G$ is called a breaking point for $L(G)$ if and only if

$$\quad \text{for every } X \in L(G), \quad \text{we have } X \leq H \text{ or } X \geq H.$$  

Such subgroups have been studied in paper \cite{1}. In paper \cite{5}, the author extended the concept to the poset of cyclic subgroups of a finite group, denoted by $C(G)$, and proved that the generalized quaternion 2-groups are the only finite noncyclic groups whose posets of cyclic subgroups have breaking points. Further, also in the paper \cite{5}, the author generalized the concept again and extended it to the poset of conjugacy classes of cyclic subgroups of $G$, denoted by $\mathcal{C}(G) = \{ [H] | H \in C(G) \}$. It seems that $[H]$ being a breaking point of $\mathcal{C}(G)$ is weaker than the condition where $H$ is a breaking point of $C(G)$. And the author \cite{5} remarked that for a finite $p$-group $G$, the poset $\mathcal{C}(G)$ possesses breaking points if and only if $G$ is either a cyclic $p$-group of order at least $p^2$ or a generalized quaternion 2-group, and that for an arbitrary finite group $G$, the problem of

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characterizing the existence and the uniqueness of breaking points of $\mathcal{C}(G)$ remains still open. In this note, we will answer this open problem. Our main theorem proves that the generalized quaternion 2-groups exhaust all finite noncyclic groups whose posets of conjugacy classes of cyclic subgroups have breaking points.

**Theorem 1.1.** Let $G$ be a finite group. Then the poset $\mathcal{C}(G)$ possesses breaking points if and only if $G$ is either a cyclic $p$-group of order at least $p^2$ or a generalized quaternion 2-group.

Further, by Theorem 1.1 of [5], we can obtain that for a finite group $G$, two conditions, i.e., the poset $\mathcal{C}(G)$ has breaking points and the poset $\mathcal{C}(H)$ has breaking points, are equivalent.

The notation and terminologies are standard in this note, and the reader is referred to [3] for group theory and [4] for subgroup lattice theory if necessary.

2. The proof of Theorem 1.1

To prove the Theorem 1.1, we cite the following crucial Theorem 1 of paper [2], proved by using the classification of finite simple groups.

**Theorem 2.1.** Let $G$ be a finite group acting transitively on a set $\Omega$ with $|\Omega| > 1$. Then there exists a prime $r$ and an $r$-element $g \in G$ such that $g$ acts without fixed points on $\Omega$.

As we all know, a finite group can be generated by the representatives of all its conjugacy classes. Using Theorem 2.1, we can generalize this conclusion and obtain that a finite group can be generated by the representatives of all its conjugacy classes of prime power order elements. This is the following lemma.

**Lemma 2.2.** Let $G$ be a finite group and $H$ be a subgroup of $G$. Suppose that for each prime power order element, there exists some element $g \in G$ such that $x^g \in H$. Then $G = H$.

**Proof.** By the way of contradiction, assume that $H$ is a proper nontrivial subgroup of $G$. Let $\Omega$ be the set of right cosets of $H$ in $G$. Then $\Omega = \{Hg | g \in G\}$ and $|\Omega| > 1$. Considering the action of $G$ on the set $\Omega$, we have that $G$ acts transitively on $\Omega$, and so for each element $Hg \in \Omega$, the stabilizer $G_{Hg}$ of $Hg$ is equal to $H^g$. By hypothesis, since every element of prime power order of $G$ is conjugate to an element of $H$, we get that each element of prime power order of $G$ has a fixed point on the set $\Omega$. On the other hand, in view of $|\Omega| > 1$ and $G$ acting transitively on $\Omega$, by Theorem 2.1 we have that there exists a prime power order element that acts fixed-point-free on $\Omega$. Hence a contradiction is derived, and thus $G = H$. □

For convenience, we put a remark of the paper [5] about a finite $p$-group as the next lemma.

**Lemma 2.3.** Let $G$ be a finite $p$-group. Then the poset $\mathcal{C}(G)$ possesses breaking points if and only if $G$ is either a cyclic $p$-group of order at least $p^2$ or a generalized quaternion 2-group.

**Proof of Theorem 1.1.** Since the necessity is obvious, it is enough to prove the sufficiency. And by Lemma 2.3, it is sufficient to prove that $G$ must be a group of prime power order.

Assume that $G$ is not a group of prime power order. Then $|\pi(G)| > 1$, that is, $|G|$ has at least two distinct prime divisors. Let $[H]$ be a breaking point of $\mathcal{C}(G)$. By the definition of $\mathcal{C}(G)$, we have that for any $X \in \mathcal{C}(G)$, there exists an element $g \in G$ satisfying that $X^g \leq H$ or $X^g \geq H$. It follows that $|H|$ has more than two distinct prime divisors. Let $p \in \pi(G)$ and $K$ be a cyclic $p$-subgroup of $G$. Then there exists an element $g \in G$ such that $K^g \leq H$ or $K^g \geq H$. Since $|\pi(H)| > 1$, we get $K^g \leq H$. Hence for every prime power order element $x \in G$, $x$ is conjugate to an element of $H$. By Lemma 2.2, we have $G = H$, a contradiction with $H < G$. Therefore, $G$ is a group of prime power order. □

By the results of Theorem 1.1, we easily obtain the following two corollaries.

**Corollary 2.4.** Let $G$ be a finite group. Then the poset $\mathcal{C}(G)$ possesses a unique breaking point if and only if $G$ is either a cyclic $p$-group of order $p^2$ or a generalized quaternion 2-group.

**Corollary 2.5.** The generalized quaternion 2-groups are the only finite noncyclic groups whose posets of conjugacy classes of cyclic subgroups have breaking points.

Comparing Theorem 1.1 with Theorem 1.1 of [5], we obtain the following corollary.

**Corollary 2.6.** Let $G$ be a finite group. Then the poset $\mathcal{C}(G)$ has breaking points if and only if the poset $\mathcal{C}(G)$ has breaking points.
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