# Symmetries of the backward heat equation with potential and interest rate models 

# Symétries de l'équation de la chaleur rétrograde avec potentiel et modèles de taux d'intérêt 

Paul Lescot, Hélène Quintard<br>Normandie Université, Université de Rouen, Laboratoire de Mathématiques Raphaël-Salem, CNRS, UMR 6085, avenue de l'Université, BP 12, 76801 Saint-Étienne-du-Rouvray cedex, France

## A R T I C L E IN F O

## Article history:

Received 19 December 2013
Accepted after revision 11 March 2014
Available online 29 April 2014
Presented by the Editorial Board


#### Abstract

We compute the isovector algebra of the Hamilton-Jacobi-Bellman equation when the potential belongs to a class that strictly includes quadratic potentials, and then determine a canonical basis for it. This setting allows us to parameterize canonically the important class of one factor interest rate models.


(C) 2014 Published by Elsevier Masson SAS on behalf of Académie des sciences.

## R É S U M É

Nous calculons l'algèbre des isovecteurs de l'équation de Hamilton-Jacobi-Bellman lorsque le potentiel appartient à une certaine classe, qui inclut strictement celle des potentiels quadratiques, et en déterminons ensuite une base canonique. Ce cadre nous permet de paramétrer canoniquement l'importante classe des modèles affines de taux d'intérêt à un facteur.
© 2014 Published by Elsevier Masson SAS on behalf of Académie des sciences.

## 1. Introduction

We determine the Lie algebra $\mathcal{H}_{(C, D)}$ of pure isovectors for the backward heat equation

$$
\begin{equation*}
\gamma^{2} \frac{\partial \eta}{\partial t}=-\frac{\gamma^{4}}{2} \frac{\partial^{2} \eta}{\partial q^{2}}+V \eta \tag{V}
\end{equation*}
$$

or more precisely for the related (HJB) equation $\left(\mathcal{E}_{V}^{\prime}\right)$ obtained by setting $S=-\gamma^{2} \ln (\eta)$, with potential $V=\frac{C}{q^{2}}+D q^{2}$; this generalizes previously known results [5,6] corresponding to $C=D=0$, and to $D=0, C>0$. It turns out that, for a fixed value of $D$, whichever $C \neq 0, \mathcal{H}_{C, D}$ is a certain 4 -dimensional subalgebra $\mathcal{P}_{D}$ of $\mathcal{H}_{0, D}$, which is itself 6 -dimensional; furthermore, $\mathcal{P}_{D}$ and $\mathcal{H}_{0, D}$ possess canonical bases continuous in $D$ and compatible with the inclusions $\mathcal{P}_{D} \subset \mathcal{H}_{0, D}$.

[^0]This potential appears naturally in the study of one-factor interest rate models: each such model can be parameterized by a Bernstein process for $V$.

The contents of Section 4 first appeared in the second author's Master's Thesis (Rouen, 2011).
The complete proofs of these results will appear along with other material in [7].

## 2. Context and previous results about the heat equation with potential

First we need to recall some definitions and properties of some useful tools. For more details, the reader could refer to [2] and [1].

Let $E$ be a vector space of dimension $n$, and $U$ an open subset of $E$.
Definition 2.1. A differential form of degree $p, p \in \mathbb{N}$ on $U$ is a smooth application from $U$ to $\bigwedge^{p} E^{*}$ where $\bigwedge^{p} E^{*}$ is the space of linear alternating $p$-forms on $E$. We call $\Omega^{p}(U)$ the set of those forms and we set $\Omega(U)=\bigoplus_{p=0}^{\infty} \Omega^{p}(U)$.

Theorem 2.2. There is a unique linear application $\mathrm{d}: \Omega(U) \rightarrow \Omega(U)$ with the following properties:
(1) $\mathrm{d}\left(\Omega^{p}(U)\right) \subset \Omega^{p+1}(U)$;
(2) on $\Omega^{0}(U)$, d is the differential on functions;
(3) if $\alpha, \beta \in \Omega(U), \mathrm{d}(\alpha \wedge \beta)=\mathrm{d} \alpha \wedge \beta+(-1)^{\operatorname{deg}(\alpha)} \alpha \wedge \mathrm{d} \beta$;
(4) $d \circ d=0$.

This application is called the exterior differential.
Definition 2.3. Let $U$ and $V$ be two open subsets of vector spaces $M$ and $N$ and $f \in \mathcal{C}^{\infty}(U, V)$. The inverse image by $f$ of $\alpha \in \Omega(V)$, noted $f^{*} \alpha$, is the form on $V$ defined by: $\left(f^{*} \alpha\right)_{x}={ }^{t}\left(T_{x} f\right) \cdot \alpha_{f(x)}$, where $T_{x} f$ is the linear tangent application to $f$ at point $x$.

Definition 2.4. The Lie derivative associated with a vector field $X$ is the linear application $\mathcal{L}_{X}: \Omega^{p}(U) \rightarrow \Omega^{p}(U), \alpha \mapsto$ $\left(\frac{\mathrm{d}}{\mathrm{d} t}\right)\left(\phi_{t}^{*} \alpha\right)_{\mid t=0}$ where $\phi_{t}$ is the local one-parameter group associated with $X$.
$\phi_{t}^{*} \alpha$ is a differential form.
Theorem 2.5. The operator $\mathcal{L}_{X}$ is characterized by the following properties:
(1) if $f \in \mathcal{C}^{\infty}(U), \mathcal{L}_{X} f=\mathrm{d} f(X)=X \cdot f$;
(2) $\mathcal{L}_{X} \circ \mathrm{~d}=\mathrm{d} \circ \mathcal{L}_{X}$;
(3) for all $\alpha$ and $\beta$ differential forms we have $\mathcal{L}_{X}(\alpha \wedge \beta)=\mathcal{L}_{X} \alpha \wedge \beta+\alpha \wedge \mathcal{L}_{X} \beta$ (i.e. $\mathcal{L}_{X}$ is a derivation of the algebra $\Omega(U)$ ).

Now we recall the steps of the Harrison-Estabrook method to find the symmetries of a PDE system (E) of order $p$ on $U$. We will consider every partial derivative of order $i, i<p$ as a new variable (and call $m$ the number of those new variables), that gives a certain number of new PDEs. The differential system of all those PDEs and (E) will be called ( $E^{\prime}$ ). Then we find a family of differential forms such that their vanishing is equivalent to $\left(\mathrm{E}^{\prime}\right)$.

We call $I$ the closed ideal spanned by those differential forms. The goal is to find a submanifold of dimension $n$ in the space of dimension $m+n$ called $M$ where the differential forms vanish.

Definition 2.6. An isovector $N$ of $(\mathrm{E})$ is a vector field on $M$ such that $\mathcal{L}_{N}(I) \subset I$.
Theorem 2.7. (See [1].)
(1) The isovectors of (E) form a Lie algebra.
(2) This Lie algebra is independent of the new variables up to an isomorphism.

Let's study Eq. ( $\mathcal{E}_{V}$ ) with those tools.
The first step is to transform this equation in a Hamilton-Jacobi-Bellman equation. To perform that we proceed as in [6], setting $S=-\gamma^{2} \ln (\eta)$. This gives the equation:

$$
\begin{equation*}
-\frac{\partial S}{\partial t}+\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^{2}-V-\frac{\gamma^{2}}{2} \frac{\partial^{2} S}{\partial q^{2}}=0 \tag{V}
\end{equation*}
$$

We denote by $\mathcal{G}_{V}$ the isovector algebra for that equation (see [6], (3.2), p. 209).
A typical element $N$ of $\mathcal{G}_{V}$ is then of the form $N=N^{q} \frac{\partial}{\partial q}+N^{t} \frac{\partial}{\partial t}+N^{S} \frac{\partial}{\partial S}+N^{B} \frac{\partial}{\partial B}+N^{E} \frac{\partial}{\partial E}$.

The $N^{*}$ are determined by formulas (3.9') and (3.22)-(3.29) in the proof of Theorem 3.3 in [6], which can be summarized as:

$$
\begin{align*}
& N^{q}=\frac{1}{2} T_{N}^{\prime}(t) q+l(t), \quad N^{t}=T_{N}(t), \quad N^{S}=h(q, t, S), \\
& N^{B}=\frac{1}{2} T_{N}^{\prime}(t) B-\frac{\partial h}{\partial q}+B \frac{\partial h}{\partial S}, \\
& N^{E}=-\left(\frac{1}{2} T_{N}^{\prime \prime}(t) q+l^{\prime}(t)\right) B-T_{N}^{\prime}(t) E-\frac{\partial h}{\partial t}+E \frac{\partial h}{\partial S}, \tag{1}
\end{align*}
$$

where $p, l$ and $\phi$ satisfy:

$$
\begin{align*}
& h(q, t, S)=\gamma^{2} p(q, t) e^{\frac{1}{\gamma^{2}} S}-\phi(q, t), \quad \phi(q, t)=\frac{1}{4} T_{N}^{\prime \prime} q^{2}+l^{\prime} q-\sigma(t) \\
& \gamma^{2} \frac{\partial p}{\partial t}=-\frac{\gamma^{4}}{2} \frac{\partial^{2} p}{\partial q^{2}}+p V \\
& -\frac{\partial \phi}{\partial t}+T_{N} \frac{\partial V}{\partial t}+\left(\frac{1}{2} T_{N}^{\prime} q+l\right) \frac{\partial V}{\partial q}-\frac{\gamma^{2}}{2} \frac{\partial^{2} \phi}{\partial q^{2}}+T_{N}^{\prime} V=0 \tag{2}
\end{align*}
$$

We set $\mathcal{H}_{V}=\left\{N \in \mathcal{G}_{V} \left\lvert\, \frac{\partial N^{s}}{\partial S}=0\right.\right\}$ (for $V \equiv 0, \mathcal{H}_{V}$ is $\mathcal{H}$ from [5]). The elements of $\mathcal{H}_{V}$ are termed "pure isovectors".
Let $\mathrm{e}^{\mu N} \operatorname{map}(t, q, S, B, E)$ to $\left(t_{\mu}, q_{\mu}, S_{\mu}, B_{\mu}, E_{\mu}\right)$, and $\eta_{\mu}\left(t_{\mu}, q_{\mu}\right)=\mathrm{e}^{-\frac{1}{\gamma^{2}} S_{\mu}}$; then $\eta_{\mu}$ is a solution of ( $\mathcal{E}_{V}$ ). Setting $\mathrm{e}^{\mu \tilde{N}(\mu)}:=\eta_{\mu}$, we obtain:

$$
\begin{equation*}
\tilde{N} \eta(t, q)=-N^{t} \frac{\partial \eta}{\partial t}-N^{q} \frac{\partial \eta}{\partial q}-N^{S} \frac{1}{\gamma^{2}} \eta \tag{3}
\end{equation*}
$$

(see [6], p. 216, and, in the context of the Black-Scholes equation, [4]).
Lemma 2.8. $\mathcal{H}_{V}$ is a Lie subalgebra of $\mathcal{G}_{V}$ (see [5], p.214) and $N \mapsto-\tilde{N}$ is a Lie algebra morphism, in particular $\tilde{\mathcal{H}}_{V}:=\left\{\tilde{N} \mid N \in \mathcal{H}_{V}\right\}$ is a Lie algebra.

## 3. Parametrization of a one-factor affine model

In this section, we will show that the one-factor affine model can be parameterized by a Schrödinger process for $V(t, q)=$ $\frac{C}{q^{2}}+D q^{2}$.

First we recall the definition of a Schrödinger process. A very general definition was given in [9] under the name of Bernstein process. For more clarity, we shall use the definition from a more recent paper [8]:

Definition 3.1. A Schrödinger process between two Borelian probability measures $\mu_{0}, \mu_{T}$ on $\mathbb{R}$ associated with a solution $\eta$ of $\left(\mathcal{E}_{V}^{\prime}\right)$ is a process $\left(X_{t}\right)_{t \in[0, T]}$ verifying that, for $u=\gamma^{2} \frac{1}{\eta} \frac{\partial \eta}{\partial q}$ :
(1) $X_{t}=X_{0}+\int_{0}^{T} u\left(s, X_{s}\right) \mathrm{d} s+\gamma B_{t}$, where $B_{t}$ is a Brownian motion;
(2) $X_{0}$ (resp. $X_{T}$ ) has law $\mu_{0}$ (resp. $\mu_{T}$ );
(3) $\mathcal{E}\left[\int_{0}^{T}\left[[1 / 2]\left|u\left(t, X_{t}\right)\right|^{2}+V\left(t, X_{t}\right)\right] \mathrm{d} t\right]<\infty$.

According to Theorems 2.2 and 2.4 of [8], if we take $\mu_{0}$ and $\mu_{T}$ two Borelian probability measures on $\mathbb{R}$, there exists a unique solution $\eta$ of ( $\mathcal{E}_{V}^{\prime}$ ) and a unique process $X$ that satisfy the previous definition.

We say that $\left(X_{t}\right)_{t \geq 0}$ is a Schrödinger process if, for all $T>0,\left(X_{t}\right)_{t \in[0, T]}$ is a Schrödinger process on [0,T] for the law $\mu_{0}$ of $X_{0}$ and the law $\mu_{T}$ of $X_{T}$.

A one-factor affine interest rate model is characterized by the instantaneous rate $r(t)$, satisfying the following stochastic differential equation:

$$
\begin{equation*}
\mathrm{d} r(t)=\sqrt{\alpha r(t)+\beta} \mathrm{d} w(t)+(\phi-\lambda r(t)) \mathrm{d} t \tag{4}
\end{equation*}
$$

under the risk-neutral probability $Q$ ( $\alpha=0$ corresponds to the Vasicek model, and $\beta=0$ corresponds to the Cox-IngersollRoss model; cf. [3]).

Assuming $\alpha>0$, let us set $\tilde{\phi}={ }_{\operatorname{def}} \phi+\frac{\lambda \beta}{\alpha}$ and $\delta=_{\operatorname{def}} \frac{4 \tilde{\phi}}{\alpha}$, and we also assume that $\tilde{\phi} \geq 0$. Let us then set $X_{t}=\alpha r(t)+\beta$ and $z(t)=\sqrt{X_{t}}$.

Theorem 3.2. Consider the stopping time $T=\inf \left\{t>0 \mid X_{t}=0\right\} ; T=+\infty$ a.s. for $\delta \geq 2$, and $T<+\infty$ a.s. for $\delta<2$, and there exists a Schrödinger process $y(t)$ for $\gamma=\frac{\alpha}{2}$ and the potential $V(t, q)=\frac{c}{q^{2}}+D q^{2}$, where $C:=\frac{\alpha^{2}}{8}\left(\tilde{\phi}-\frac{\alpha}{4}\right)\left(\tilde{\phi}-\frac{3 \alpha}{4}\right)=\frac{\alpha^{4}}{128}(\delta-1)(\delta-3)$, and $D:=\frac{\lambda^{2}}{8}$ such that $\forall t \in[0, T[z(t)=y(t)$. In particular, for $\delta \geq 2, z$ itself is a Bernstein process.

## 4. Isovectors in the case $V(t, q)=\frac{C}{q^{2}}+D q^{2}$

Theorem 3.2 motivates us to find the explicit form of the isovectors for the potential: $V(t, q)=\frac{C}{q^{2}}+D q^{2}$.
We shall denote $\mathcal{H}_{(C, D)}=\mathcal{H}_{V}, \mathcal{P}_{D}=\mathcal{H}_{(0, D)}, \tilde{\mathcal{H}}_{(C, D)}=\tilde{\mathcal{H}}_{V}$ and $\tilde{\mathcal{P}}_{D}=\tilde{\mathcal{H}}_{(0, D)}$.
Theorem 4.1. For $C \neq 0, \mathcal{H}_{(C, D)}=\mathcal{H}_{(1, D)} \simeq \mathcal{H}_{(1,0)}$ has dimension 4; for $C=0, \mathcal{H}_{(C, D)} \simeq \mathcal{H}_{(0,0)}$ has dimension 6. Furthermore, these Lie algebras possess canonical bases, continuous in $D$ for a fixed value of $C$, and compatible with the inclusions $\mathcal{H}_{(C, D)} \subseteq \mathcal{H}_{(0, D)}$.

Corollary 4.2. The isovector algebra $\mathcal{H}_{V}$ associated with $V$ has dimension 6 if and only if $\tilde{\phi} \in\left\{\frac{\alpha}{4}, \frac{3 \alpha}{4}\right\}$, i.e. $\delta \in\{1$, 3\}; in the opposite case, it has dimension 4.

In particular, the isovector algebra associated with the affine model has dimension 6 if and only if $\delta \in\{1,3\}$; in the opposite case, it has dimension 4.

## References

[1] B.K. Harrison, F.B. Estabrook, Geometric approach to invariance groups and solution of partial differential systems, J. Math. Phys. 12 (1971) 653-666.
[2] J. Lafontaine, Introduction aux variétés différentielles, Collection Grenoble Sciences, Presses universitaires de Grenoble, 1996.
[3] B. Leblanc, O. Scaillet, Path-dependent options on yields in the affine term structure model, Finance Stoch. 2 (4) (1998) 349-367.
[4] P. Lescot, Symmetries of the Black-Scholes equation, Methods Appl. Anal. 19 (2012) 147-160.
[5] P. Lescot, J.-C. Zambrini, Isovectors for the Hamilton-Jacobi-Bellman equation, formal stochastic differentials and first integrals in Euclidean quantum mechanics, in: Seminar on Stochastic Analysis, Random Fields and Applications IV, in: Progr. Probab., vol. 58, Birkhäuser, Basel, 2004, pp. 187-202.
[6] P. Lescot, J.-C. Zambrini, Probabilistic deformation of contact geometry, diffusion processes and their quadratures, in: Seminar on Stochastic Analysis, Random Fields and Applications V, in: Progr. Probab., vol. 59, Birkhäuser, Basel, 2008, pp. 203-226.
[7] P. Lescot, H. Quintard, J.-C. Zambrini, Solving stochastic differential equations with Cartan's exterior differential system, in preparation.
[8] M. Thieullen, J.-C. Zambrini, Symmetries in the stochastic calculus of variation, Probab. Theory Relat. Fields 107 (1997) 401-427.
[9] J.-C. Zambrini, Euclidean quantum mechanics, Phys. Rev. A 35 (9) (1987) 3630-3649.


[^0]:    E-mail addresses: paul.lescot@univ-rouen.fr (P. Lescot), helene.quintard@gmail.com (H. Quintard).
    http://dx.doi.org/10.1016/j.crma.2014.03.024
    1631-073X/© 2014 Published by Elsevier Masson SAS on behalf of Académie des sciences.

