Note on the differentiability of arithmetic Fourier series arising from Eisenstein series

Note sur la dérivabilité de séries arithmétiques de Fourier provenant des séries d’Eisenstein

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1. Introduction and statement of the results

In this note, we present some results concerning the differentiability of certain Fourier series related to Eisenstein series. Let \( k \in \mathbb{N}^* \) be even. Recall that the Eisenstein series of weight \( k \) over the upper-half plane \( \mathbb{H} \) is defined by the first equality

\[
E_k(z) = \frac{1}{2 \zeta(k)} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m+nz)^k} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi inz},
\]

The aim of this note is to present results concerning the differentiability of some Fourier series arising from Eisenstein series. Sine series exhibit different behaviours with respect to differentiability than the series with cosine function. The precise results are given for the series related to Eisenstein series of weight 2, whereas for the series arising from Eisenstein series of higher weight we conjecture the results.

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and the second equality provides its Fourier series, where $B_k$ is the $k$-th Bernoulli number, and $\sigma_k(n) = \sum_{d|n} d^{k-1}$. If $k = 2$, the function is quasi-modular under the action of $SL_2(\mathbb{Z})$, and for $k \geq 4$ it is modular. We are interested in the real-valued continuous functions $F_k$, $G_k$ defined by

$$F_k(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1}} \sin(2\pi nx) \quad \text{and} \quad G_k(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1}} \cos(2\pi nx).$$

In particular, we focus on the differentiability of $F_2$ and $G_2$. The differentiability of $F_2$ at $x$ depends on the continued fraction expansion of $x$, but $G_2$ is probably differentiable at any $x \in \mathbb{R} \setminus \mathbb{Q}$. Already in 1933, Wilton in his work [7] proved that there is a connection between certain trigonometric series involving the divisor function $\sigma_0(n)$ and continued fractions.

**Theorem 1.3.** Neither $F_2$ nor $G_2$ is differentiable at any $x \in \mathbb{Q}$. However, $G_2$ is right and left differentiable at each $x \in \mathbb{Q}$.

Let $(\frac{p_n(x)}{q_n(x)})_{n \geq 0}$ be the sequence of continued fraction approximations of $x$, i.e. $\frac{p_n(x)}{q_n(x)} = [a_0; a_1, a_2, \ldots, a_n]$. We then make the following definition, which is motivated by the work of Brjuno [1].

**Definition 1.2.** Let $x \in \mathbb{R} \setminus \mathbb{Q}$. We will say that $x$ is a square-Brjuno number if $\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n(x)^2} < \infty$.

In addition, we introduce two technical conditions:

$$\lim_{n \to \infty} \frac{\log q_{n+4}(x)}{q_n(x)^2} = 0; \quad \lim_{n \to \infty} \frac{\log q_{n+3}(x)}{q_n(x)^2} = 0, \quad \text{and} \quad a_n = 1 \text{ for only finitely many } n. \quad (1)$$

We note that the square-Brjuno property and conditions (1) and (2) are independent.

**Theorem 1.3.**

(i) If $x \in \mathbb{R} \setminus \mathbb{Q}$ is a square-Brjuno number satisfying (1) or (2), then $F_2$ is differentiable at $x$. If $x \in \mathbb{R} \setminus \mathbb{Q}$ is not a square-Brjuno number, then $F_2$ is not differentiable at $x$.

(ii) Let $x \in \mathbb{R} \setminus \mathbb{Q}$ satisfy (1) or (2), then $G_2$ is differentiable at $x$.

We observe that condition (1) is satisfied for almost all $x$, but that condition (2) holds for almost no $x$. We believe that both conditions (1) and (2) could be removed in Theorem 1.3; however, the applied method does not enable us to do this. Moreover, almost all numbers are square-Brjuno. If $x \in \mathbb{R} \setminus \mathbb{Q}$ is not square-Brjuno, then it must be Liouville. We prove Theorems 1.1 and 1.3 using the method proposed by Itatsu in the study of the Riemann “non-differentiable” function $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\pi n^2 x)$, see [3].

2. Sketch of the proofs

Let $\varphi_2$ be the complex-valued function defined by $\psi_2(x) = G_2(x) + iF_2(x)$. For a matrix $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z})$ and $z \in \mathbb{C}$, we will denote the fraction transformation as $\gamma \cdot z = \frac{az + b}{cz + d}$, if $cz + d \in \mathbb{C} \setminus \{0\}$, and $\gamma \cdot (-\frac{z}{d}) = \infty$. Recall that for all $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z})$ and all $z \in \mathbb{H}$, we have $E_2(z) = E_2(\gamma \cdot z)(cz + d)^{-2} + 6ic\pi^{-1}(cz + d)^{-1}$. Based on this quasi-modular identity and the relationship between $\varphi_2$ and $E_2$, we obtained the following functional equation for $\varphi_2$.

**Lemma 2.1.** Let $\frac{p}{q} \in \mathbb{Q}$, $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z})$, and $x \in \mathbb{R}$. We have

$$\varphi_2(x) = (qx - p)^4 \varphi_2(\gamma \cdot x) - \frac{1}{3q^2} (qx - p) \log(qx - p) + P_\gamma(qx - p)$$

$$- \frac{\pi^2}{q^2} (qx - p)^2 \log(qx - p) + 6 \int_0^x q(qt - p)^2(q(x - t) - (qt - p)) \varphi_2(\gamma \cdot t) \, dt, \quad (3)$$

where $\log$ denotes the principal value of the complex logarithm and $P_\gamma \in \mathbb{C}[x]$ is a polynomial of degree less than or equal to 3 that depends on $\gamma$. 

We then take the imaginary (or real) part of each side of Eq. (3). The conclusion of Theorem 1.1 follows from letting $x \to \frac{p}{q}$ and estimating the growth rates of each of the terms of (3). For $F_2$, the principal term is given by $F_2(x) = F_2(\frac{p}{q}) = \frac{1}{3p} (q(x-p)) \log |q(x-p)(1 + o(1))|$, which explains that $F_2$ is not differentiable at $\frac{p}{q}$, whereas for $G_2$ the term $\text{Re}(-\frac{i}{3p} (q(x-p)) \log |q(x-p)|)$ determines the differentiability properties at $\frac{p}{q}$.

Let $T : [0, 1) \to [0, 1)$ denote the Gauss map, that is $T(0) = 0$ and $T(x) = \frac{1}{x} \mod 1$ if $x \neq 0$. The function $T$ is differentiable at all $x \notin \mathbb{Q}$. We apply (3) to the matrix $G = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$. Since $\varphi_2$ is 1-periodic, we have $\varphi_2(-\frac{1}{x}) = \varphi_2(-T(x))$. By taking the imaginary part of each side of the equation, we deduce from (3) the following equation.

**Lemma 2.2.** For all $x \in [0, 1)$ we have

$$F_2(x) = -x^A F_2(T(x)) - \frac{\pi^2}{3} x \log(x) + P(x) - 6 \int_0^x t^2(x - 2t) F_2(T(t)) \, dt,$$

where $P \in \mathbb{R}[x]$ is of degree less than or equal to 3.

For brevity, write $\beta_1(x) = 1$ and $\beta_n(x) = \prod_{j=0}^{n-1} T^j(x)$ for $n \geq 0$. Assume $x \notin \mathbb{Q}$. We then iterate Eq. (4) repeatedly substituting $T(x)$ for $x$ to obtain the following.

**Lemma 2.3.** For all $x \in [0, 1) \setminus \mathbb{Q}$ and all $n \in \mathbb{N}^*$ we have

$$F_2(x) = (-1)^n F_2(T^n(x)) \beta_{n-1}(x)^4 + 6 \sum_{j=1}^{n} (-1)^j \beta_{j-2}(x)^4 \int_0^{T^{j-1}(x)} t^2(T^{j-1}(x) - 2t) F_2(T(t)) \, dt$$

$$+ \frac{\pi^3}{3} \sum_{j=1}^{n} (-1)^j \log(T^{j-1}(x)) \beta_{j-1}(x) \beta_{j-2}(x)^2 + \sum_{j=1}^{n} (-1)^{j-1} P(T^j(x)) \beta_{j-2}(x)^4,$$

where $P \in \mathbb{R}[x]$ is the same polynomial as in Lemma 2.2.

If $n \to \infty$, the three series in (5) converge for all $x \in \mathbb{R}$, and the term $(-1)^n F_2(T^n(x)) \beta_{n-1}(x)^4 \to 0$, which provides an alternative expression for $F_2$.

We are interested in the limit $\lim_{h \to 0} \frac{F_2(x + h) - F_2(x)}{h}$. Therefore, for each $h > 0$, we choose a suitable $n_h \in \mathbb{N}$, which satisfies $n_h \to \infty$ and is non-decreasing as $h \to 0$. We then apply Lemma 2.3 with $n = n_h$ and analyse the rate of change of each of the terms in (5) as $h \to 0$. The next lemma illustrates the origin of the square-Brjuno condition in Theorem 1.3.

**Lemma 2.4.** Write $f'(x)$ for the derivative of a function $f$ at $x$. Given $x \in \mathbb{R} \setminus \mathbb{Q}$, the series

$$\frac{\pi^3}{3} \sum_{j=1}^{\infty} (-1)^j \left( \log(T^{j-1}(x)) \beta_{j-1}(x) \beta_{j-2}(x)^3 \right)'$$

converges if and only if $x$ is square-Brjuno. On the other hand, the two series

$$6 \sum_{j=1}^{\infty} (-1)^j \left( \beta_{j-2}(x)^4 \int_0^{T^{j-1}(x)} t^2(T^{j-1}(x) - 2t) F_2(T(t)) \, dt \right)'$$

and

$$\sum_{j=1}^{\infty} (-1)^{j-1} P(T^j(x)) \beta_{j-2}(x)^4 \right)'$$

converge for all $x \in \mathbb{R} \setminus \mathbb{Q}$.

The technical conditions (1) and (2) arise from the analysis of the term

$$\frac{1}{n} \left( (-1)^n F_2(T^n(x + h)) \beta_{n-1}(x + h)^4 - (-1)^n F_2(T^n(x)) \beta_{n-1}(x)^4 \right).$$

As $h \to 0$, it converges to 0 if either (1) or (2) is satisfied.

We follow the same strategy for $G_2$, however, we do not have an analogue of (6) which explains why we only need (1) or (2) in Theorem 1.3(ii). The detailed proofs of Theorems 1.1 and 1.3 will be published elsewhere [5]. If $F_2$ or $G_2$ is differentiable at $x$ by Theorem 1.3, the value of the derivative is given by the addition of (6) and (7).
3. Generalisation

**Conjecture 3.1.** Let \( k \in \mathbb{N}^* \) be even. We have the following.

(i) Neither \( F_k \) nor \( G_k \) is differentiable at any \( x \in \mathbb{Q} \). However, \( G_k \) is right and left differentiable at each \( x \in \mathbb{Q} \).
(ii) The function \( G_k \) is differentiable at any \( x \in \mathbb{R} \setminus \mathbb{Q} \).
(iii) The function \( F_k \) is differentiable at \( x \in \mathbb{R} \setminus \mathbb{Q} \) if and only if

\[
\sum_{n=0}^{\infty} \frac{\log q_{n+1}(x)}{q_n(x)^k} < \infty. \tag{8}
\]

In order to prove Conjecture 3.1 for \( k \geq 4 \), we would proceed as in the case \( k = 2 \). There are a lot of terms to analyse, but we believe that for any given \( k \geq 4 \), the method presented here would work (adding a technical condition similar to (1) of the type \( \frac{\log(q_{n+1})}{q_n} \rightarrow 0 \)). However, the calculations become very long, and we do not do it explicitly. In [5], we present arguments justifying the conjecture.

Chamizo [2] studied the differentiability of series arising from automorphic forms \( f(x) = \sum_{n=0}^{\infty} r_n e^{2\pi i nx} \) of positive weights \( k \) under the Fuchsian group with a multiplier system: \( f_s(x) = \sum_{n=1}^{\infty} r_n s^n e^{2\pi i nx} \). His method is based on the theory of automorphic forms. However, he requires \( s < \frac{k}{2} + 1 \) for \( f \) not being a cusp form. In our case, \( s = k + 1 \), which does not enable us to apply his results. Another method of analysing such series involves wavelet methods, and was proposed by Jaffard [4], again, in the study of \( S(x) \). Studying the Hölder regularity exponent of \( F_k \) and \( G_k \) with this method enables us to prove some cases of Conjecture 3.1. For each \( n \), we define \( \kappa_n \) by the equality \( |x - \frac{p_n}{q_n}| = \frac{1}{q_n^\kappa_n} \), and we let \( \mu(x) = \lim \sup_{n \to \infty} \kappa_n \).

\( v(x) = \lim \inf_{n \to \infty} \kappa_n \). It has been proved in [6] that for \( k \geq 4 \) and \( x \in \mathbb{R} \setminus \mathbb{Q} \), if \( \frac{1}{v(x)} - \frac{1}{\mu(x)} \leq \frac{1}{k} \), then the Hölder regularity exponents of \( F_k \) and \( G_k \) at \( x \) are both \( 1 + \frac{k}{\mu(x)} \). If \( \mu(x) < \infty \), then both \( F_k \) and \( G_k \) are differentiable at \( x \). The condition \( \mu(x) < \infty \) implies (8), and we see that one direction of Conjecture 3.1(iii) is true. Since for almost all \( x \), \( \mu(x) = v(x) = 2 \), the conjecture is proved for almost all \( x \) for all \( k \geq 4 \).

**References**