Ordinary differential equations/Partial differential equations

# On a $p$-Kirchhoff problem involving a critical nonlinearity 

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## A R T I CLE IN F O

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#### Abstract

This paper deals with a $p$-Kirchhoff type problem involving the critical Sobolev exponent. Under some suitable assumptions, we show the existence of at least one solution.


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## R É S U M É

On s'intéresse dans cet article au problème de p-Kirchhoff à exposant critique. On montre l'existence d'au moins une solution sous des hypothèses adéquates.
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## 1. Introduction

In the last years, mathematical models involving $p$-Kirchhoff-type equations have been extensively studied by many authors, who have proposed different methods to analyze the question of the existence of solutions and of related qualitative properties (see, for example, $[1-6,8,9,12,13]$ and references therein). For the physical and biological meaning of Kirchhofftype problems, we refer to [7,14-16].

Inspired by $[2,13]$ and by the above-mentioned papers, we consider the following problem:

$$
\begin{equation*}
h\left(\int_{\Omega}|\nabla u|^{p} \mathrm{~d} x\right)\left(-\Delta_{p} u\right)=f(x, u)+|u|^{p^{*}-2} u, \quad x \in \Omega, u=0, x \in \partial \Omega \tag{1.1}
\end{equation*}
$$

where $\Omega \subset \mathbb{R}^{N}$ is a bounded domain with smooth boundary $\partial \Omega,-\Delta_{p} u:=-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is called $p$-Laplacian, $1<p<N, p^{*}=N p /(N-p)$ is the critical exponent in the Sobolev embedding. The functions $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $h:(0, \infty) \rightarrow(0, \infty)$ are assumed to be continuous and satisfy the following conditions:
$\left(H_{1}\right) h \in L^{1}(0, s), s>0$;
$\left(H_{2}\right) \lim \sup _{t \rightarrow 0^{+}} \frac{H(t)}{t^{\alpha}}<\infty$, with $\alpha>\frac{p^{*}}{p}$, where $H(t)=\int_{0}^{t} h(s) \mathrm{d} s$;
$\left(\mathrm{H}_{3}\right)$ there exist $0<\beta \leqslant 1$ and a positive constant $C$ such that

$$
H(t) \geqslant C t^{\beta} \quad \text { for } t>0
$$

$\left(F_{1}\right)$ there exists a positive constant $C_{1}>0$ such that

$$
|f(x, t)| \leqslant C_{1}\left(1+|t|^{q-1}\right),
$$

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where $p<q<p^{*}$;
$\left(F_{2}\right)$ there exist $A>0$ and $0<\delta<\alpha p$ such that
$$
F(x, t) \geqslant A|t|^{\delta}, \quad \text { as } t \rightarrow 0,
$$
with $F(x, t)=\int_{0}^{t} f(x, s) \mathrm{d}$.
So the main result of the paper reads:

Theorem 1.1. If $\left(H_{1}\right),\left(H_{2}\right),\left(H_{3}\right)$ and $\left(F_{1}\right),\left(F_{2}\right)$ hold true, then the problem (1.1) has a nontrivial solution.
Throughout this paper, by (weak) solutions of (1.1) we understand the critical points of the associated energy functional $\phi$ acting on the Sobolev space $W_{0}^{1, p}(\Omega)$,

$$
\phi(u)=\frac{1}{p} H\left(\|u\|^{p}\right)-\int_{\Omega} F(x, u) \mathrm{d} x-\frac{1}{p^{*}} \int_{\Omega}|u|^{p^{*}} \mathrm{~d} x
$$

where $W_{0}^{1, p}(\Omega)$ is the Sobolev space endowed with the norm $\|u\|=\left(\int_{\Omega}|\nabla u|^{p} \mathrm{~d} x\right)^{\frac{1}{p}}$.
We recall that there is a continuous embedding from $W_{0}^{1, p}(\Omega)$ into $L^{r}(\Omega)$ for all $r \in\left[1, p^{*}\right]$, which implies that $\phi \in C^{1}\left(W_{0}^{1, p}(\Omega), \mathbb{R}\right)$, and then

$$
\left\langle\phi^{\prime}(u), u\right\rangle=h\left(\int_{\Omega}|\nabla u|^{p} \mathrm{~d} x\right)\left(\int_{\Omega}|\nabla u|^{p} \mathrm{~d} x\right)-\int_{\Omega} f(x, u) u \mathrm{~d} x-\int_{\Omega}|u|^{p^{*}} \mathrm{~d} x
$$

We will denote by $\lambda>0$ the best Sobolev constant of the embedding $W^{1, p}(\Omega) \subset L^{p^{*}}(\Omega)$, that is:

$$
\lambda=\inf _{u \in W^{1, p}(\Omega) \backslash\{0\}} \frac{\int_{\Omega}|\nabla u|^{p} \mathrm{~d} x}{\left(\int_{\Omega}|u|^{p^{*}} \mathrm{~d} x\right)^{\frac{p}{p^{*}}}} .
$$

## 2. Proof of the main result

The proof of our result relies on the variational principle of Ekeland; it allows us to split the geometry of the problem from the compactness aspect. Hereafter, we need some auxiliary lemmas.

Lemma 2.1. There exist $\gamma, \rho>0$ such that $\phi \geqslant \gamma$ for $\|u\|=\rho$.
Proof. By $\left(\mathrm{H}_{3}\right)$, for $\|u\|$ is sufficiently small we have,

$$
\begin{aligned}
\phi(u) & \geqslant \frac{C}{p}\|u\|^{\beta p}-\int_{\Omega} F(x, u) \mathrm{d} x-\frac{1}{p^{*}} \int_{\Omega}|u|^{p^{*}} \mathrm{~d} x \\
& \geqslant \frac{C}{p}\|u\|^{\beta p}-|\Omega|^{\frac{p^{*}-q}{p^{*}}}\left(\int_{\Omega}|u|^{p^{*}}\right)^{\frac{q}{p^{*}}}-\frac{1}{p^{*}} \lambda^{\frac{-p^{*}}{p}}\|u\|^{p^{*}}+c \\
& \geqslant \frac{C}{p}\|u\|^{\beta p}-|\Omega|^{\frac{p^{*}-q}{p^{*}}} \lambda^{\frac{-q}{p}}\|u\|^{q}-\frac{1}{p^{*}} \lambda^{\frac{-p^{*}}{p}}\|u\|^{p^{*}}+c,
\end{aligned}
$$

since $\beta p<q<p^{*}$ then there exit $\gamma, \rho>0$ such that $\phi(u)>\gamma$ for $\|u\|=\rho$.
By $\left(F_{1}\right)$ and $\left(H_{1}\right)$ we have the following remark.

## Remark 2.1.

- The functional $H$ is continuous in $[0, \infty)$ and of class $C^{1}$ on $(0, \infty)$ where $H(0)=0$.
- The functional $\phi \in C^{1}\left(W_{0}^{1, p}(\Omega)\right)$ with $\phi(0)=0$.
- Some works (e.g. [1,2,13]) assume that $h(t) \geqslant h_{0}>0$ for $t \geqslant 0$, which is not necessary for this work.

Lemma 2.2. The functional $\phi$ is bounded from below in $\bar{B}_{r}(0)$, where $\bar{B}_{r}(0)=\left\{u \in W_{0}^{1, p}(\Omega):\|u\| \leqslant r\right\}$; moreover $\widetilde{m}=\inf _{\bar{B}_{r}(0)} \phi<0$.

Proof. According to the definition of $\phi$, it is clear that the functional $\phi$ is bounded from below in $\bar{B}_{r}(0)$. Besides, let $v \in C_{0}^{\infty}(\Omega) \backslash\{0\}$ and $t>0$. Using $\left(F_{2}\right)$ and $\left(H_{2}\right)$, it follows that

$$
\begin{aligned}
\phi(t v) & =\frac{1}{p} H\left(t^{p}\|v\|^{p}\right)-\int_{\Omega} F(x, t v) \mathrm{d} x-\frac{1}{p^{*}} \int_{\Omega}|t v|^{p^{*}} \mathrm{~d} x \\
& \leqslant C_{1} t^{\alpha p}-C_{2} t^{p^{*}} \int_{\Omega}|v|^{p^{*}} \mathrm{~d} x-C_{3} t^{\delta} \int_{\Omega}|v|^{\delta} \mathrm{d} x<0
\end{aligned}
$$

for $t$ sufficiently small, where $C_{i}>0, i=1, \ldots, 3$.
Now, using the Ekeland variational principle (cf. [10]) to $\phi$ on $\bar{B}_{r}(0)$ endowed with distance $\tau(u, \omega)=\|u-\omega\|$, so there exists a sequence $\left(u_{n}\right)_{n} \subset \bar{B}_{r}$ such that:

$$
\phi\left(u_{n}\right) \rightarrow \inf _{\bar{B}_{r}} \phi=\tilde{m},
$$

we infer that

$$
\phi\left(u_{n}\right)-\phi(\omega) \leqslant \frac{\left\|u_{n}-\omega\right\|}{n}
$$

for all $\omega \neq u_{n}$. As $\phi$ is of class $C^{1}$ then

$$
\phi^{\prime}\left(u_{n}\right) \rightarrow 0
$$

and thus we have

$$
\phi\left(u_{n}\right) \rightarrow \tilde{m} \quad \text { and } \quad \phi^{\prime}\left(u_{n}\right) \rightarrow 0
$$

which yields that $\left(u_{n}\right)_{n}$ is a bounded $(P . S)_{\tilde{m}}$ sequence to $\phi$. Up to a subsequence, still denoted by $\left(u_{n}\right)_{n}$, we deduce that $\left(u_{n}\right)_{n}$ have a weak limit $u \in W_{0}^{1, p}(\Omega)$. That limit is a solution of problem (1.1). Indeed, $\phi^{\prime}\left(u_{n}\right)=0$, for some subsequence still denoted by $\left(u_{n}\right)_{n}$, for example, in view of El Hamidi and Rakotoson [11], then we have

$$
\begin{aligned}
& \nabla u_{n}(x) \rightarrow \nabla u(x), \quad \text { a.e. } x \in \Omega \\
& u_{n} \rightarrow u \quad \operatorname{in} L^{r}(\Omega), \quad 1<r<p^{*}
\end{aligned}
$$

and

$$
\left|u_{n}\right|^{p^{*}-2} u_{n} \rightarrow|u|^{p^{*}-2} u \quad \text { in } L^{\frac{p^{*}}{p^{*}-q}}(\Omega)
$$

Thus we get

$$
\left\langle\phi^{\prime}(u), v\right\rangle=\lim _{n \rightarrow \infty}\left\langle\phi^{\prime}\left(u_{n}\right), v\right\rangle,
$$

for any $v \in W_{0}^{1, p}(\Omega)$.
So it remains to prove that $u \neq 0$. It is well known that
$\phi\left(u_{n}\right) \rightarrow \widetilde{m}$ then

$$
\begin{aligned}
\tilde{m}+o(1) & =\phi\left(u_{n}\right) \\
& \geqslant \frac{C}{p}\left\|u_{n}\right\|^{\beta p}-C_{1}\left\|u_{n}\right\|^{p^{*}}-C_{2}\left\|u_{n}\right\|-C_{3}\left\|u_{n}\right\|^{q}
\end{aligned}
$$

with $C_{1}, C_{2}$ and $C_{3}>0$. It follows that

$$
C_{1}\left\|u_{n}\right\|^{p^{*}}+C_{2}\left\|u_{n}\right\|+C_{3}\left\|u_{n}\right\|^{q}>-\widetilde{m}+o(1)
$$

which implies that

$$
C_{1}\|u\|^{p^{*}}+C_{2}\|u\|+C_{3}\|u\|^{q}>-\tilde{m}>0
$$

consequently $u \neq 0$.
From the previous lemmas and by applying the Ekeland principle, the problem (1.1) has a nontrivial solution.
Remark 2.2. A standard argument shows that $\phi$ is coercive whether it is assumed that $\beta>\frac{N}{N-p}$ and we can see that $\phi$ is sequentially weakly lower semi-continuous, so we obtain a global minimum of $\phi$ since $W_{0}^{1, p}(\Omega)$ is a reflexive Banach space.

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