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# Nefness: Generalization to the lc case 

## Nefness : Généralisation au cas lc

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## A R T I C L E I N F O

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#### Abstract

This note is devoted to a proof of the b-nefness of the moduli part in the canonical bundle formula for an lc-trivial fibration that is lc and not klt over the generic point of the base. This result is proved in [3, §8] and [4] by using the theory of variation of mixed Hodge structure. Here we present a proof that makes use only of the theory of variation of Hodge structure and follows Ambro's proof of [2, Theorem 0.2]. © 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É Cette note se consacre à démontrer que la partie modulaire de la formule du fibré canonique pour une fibration qui est lc-triviale et non klt-triviale est b-semiample. Ce résultat est démontré dans $[3, \S 8]$ et dans [4] en utilisant des resultats très profonds concernant les variations de structure de Hodge mixte. On présente ici une preuve qui est plus élémentaire et qui suit celle de [2, théorème 0.2].


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## Version française abrégée

La formule du fibré canonique est un outil important en géométrie birationnelle pour traduire des propriétés qui sont vraies sur les variétés de type log-général en propriétés pour les variétés de dimension de Kodaira positive.

Cette note est consacrée à la preuve du théorème suivant. Pour les définitions de la fibration lc-triviale et de la formule du fibré canonique, voir [2].

Théorème 0.1 (Theorem 1.1). Soit $f:(X, B) \rightarrow Y$ une fibration lc-triviale. Alors il existe un morphisme propre et birationnel $Y^{\prime} \rightarrow Y$ avec les propriétés suivantes:
(i) $K_{Y^{\prime}}+B_{Y^{\prime}}$ est un diviseur $\mathbb{Q}$-Cartier ;
(ii) $M_{Y^{\prime}}$ est un diviseur nef $\mathbb{Q}$-Cartier, et pour tout morphisme propre et birationnel $v: Y^{\prime \prime} \rightarrow Y^{\prime}$, on a :

$$
v^{*}\left(M_{Y^{\prime}}\right)=M_{Y^{\prime \prime}}
$$

où $B_{Y^{\prime}}, M_{Y^{\prime}}$ et $M_{Y^{\prime \prime}}$ sont le discriminant et les parties modulaires des fibrations lc-triviales induites par le changement de base.

Le théorème 0.1 est une généralisation de [2, theorème 0.2 ]; la preuve présentée dans ce travail généralise la preuve dans [2]. Le résultat a été demontré dans [4, theorèmes 3.1, 3.4 et 3.9] et [5], en utilisant des résultats concernant les variations de structure de Hodge mixte. Pour demontrer qu'il existe un morphisme $Y^{\prime} \rightarrow Y$ tel que $M_{Y^{\prime}}$ est nef, il faut montrer que la partie modulaire, sous certaines conditions de régularité sur $f$, est le quotient d'un faisceau localement

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libre et semipositif. Cette propriété est demontrée dans le lemme 1.2, qui généralise [2, lemme 5.2]. La preuve du reste de l'énoncé, dans le cas des fibrations lc-triviales, se base sur l'isomorphisme naturel :

$$
\rho^{*} f_{*} \omega_{X / Y} \cong f_{*}^{\prime} \omega_{X^{\prime} / Y^{\prime}}
$$

où $f^{\prime}$ est la fibration induite par changement de base. Dans le cas des fibrations lc-triviales, on a besoin d'un resultat plus fort. Notamment, on demontre que, si $D=\sum D_{i}$ est un diviseur reduit sur $X$ et $D^{\prime}$ est sa transformée stricte, alors on a, pour tout $i$ un isomorphisme naturel :

$$
\rho^{*} R^{i} f_{*} \omega_{X / Y}(D) \cong R^{i} f_{*}^{\prime} \omega_{X^{\prime} / Y^{\prime}}\left(D^{\prime}\right)
$$

## 1. Introduction

The canonical bundle formula is an important tool in algebraic geometry used for translating properties that are true for varieties of general type in properties for varieties of non-negative Kodaira dimension.

This note is devoted to the proof of the following result. For all the definitions and preliminary results on the canonical bundle formula, we refer to [2].

Theorem 1.1. Let $f:(X, B) \rightarrow Y$ be an lc-trivial fibration. Then there exists a proper birational morphism $Y^{\prime} \rightarrow Y$ with the following properties:
(i) $K_{Y^{\prime}}+B_{Y^{\prime}}$ is a $\mathbb{Q}$-Cartier divisor;
(ii) $M_{Y^{\prime}}$ is a nef $\mathbb{Q}$-Cartier divisor and for every proper birational morphism $v: Y^{\prime \prime} \rightarrow Y^{\prime}$ we have

$$
v^{*}\left(M_{Y^{\prime}}\right)=M_{Y^{\prime \prime}}
$$

where $B_{Y^{\prime}}, M_{Y^{\prime}}$ and $M_{Y^{\prime \prime}}$ are the discriminant and the moduli parts of the lc-trivial fibrations induced by the base change.
Let $f:(X, B) \rightarrow Y$ be an lc-trivial fibration. Set:

$$
\begin{equation*}
D=-(\varphi)=r\left(K_{X / Y}+B-f^{*}\left(B_{Y}+M_{Y}\right)\right) \tag{1.1}
\end{equation*}
$$

Then we can define $\pi: \tilde{X} \rightarrow X$ as the normalization of $X$ in $k(X)\left(\sqrt[r]{1_{D}}\right)$ (see [3, §8.10.3] for the general construction). Moreover we have

$$
\pi_{*} \mathcal{O}_{\tilde{X}}=\bigoplus_{i=0}^{r} L^{-i}(\lfloor i D / r\rfloor) ; \quad \pi_{*} \omega_{\tilde{X}}=\bigoplus_{i=0}^{r} \omega_{X} \otimes L^{i}(-\lfloor i D / r\rfloor) .
$$

The Galois group of the extension $k(X) \subseteq k(X)\left(\sqrt[r]{1_{D}}\right)$ acts on $\pi_{*} \mathcal{O}_{\tilde{X}}$ by $\sqrt[r]{1_{D}} \mapsto \zeta \cdot \sqrt[r]{1_{D}}$ where $\zeta$ is an $r$-th primitive root of unity. The eigensheaf corresponding to $\zeta$ is $L^{-1}(\lfloor D / r\rfloor)$.

Let $B$ be a divisor such that $(X, B)$ is lc over the generic point of $Y$. We can suppose that $K_{X}+B$ has simple normal crossing support and set $E$ the sum of all the horizontal lc-centers of $(X, B)$ that dominate $Y$. Set $\tilde{E}=\pi^{*} E$, then

$$
\pi_{*}\left(\omega_{\tilde{X} / Y} \otimes \mathcal{O}_{\tilde{X}}(\tilde{E})\right)=\bigoplus_{i=0}^{r-1} \mathcal{O}_{X}\left(\left\lceil(1-i) K_{X / Y}-i B+E+i f^{*} B_{Y}+i f^{*} M_{Y}\right\rceil\right)
$$

The eigensheaf of $\zeta$ in $\pi_{*}\left(\omega_{\tilde{X} / Y} \otimes \mathcal{O}_{\tilde{X}}(\tilde{E})\right)$ is $\mathcal{O}_{X}\left(\left\lceil-B+E+f^{*} B_{Y}+f^{*} M_{Y}\right\rceil\right)$. Let $V$ be a non-singular model of $\tilde{X}$ and let $h: V \rightarrow Y$ be the induced morphism. Set $g: V \rightarrow X$ and $B_{V}=g^{*}\left(K_{X}+B\right)-K_{V}$. In [2, p. 245] are stated the following properties for $h:\left(V, B_{V}\right) \rightarrow Y$ :

- The field extension $k(V) / k(X)$ is Galois and its Galois group $G$ is cyclic of order $r$. There exists $\psi \in k(V)$ such that $\psi^{r}=\varphi$. A generator of $G$ acts by $\psi \mapsto \zeta \psi$, where $\zeta$ is a fixed primitive $r$-th root of unity.
- The relative $\log$ pair $h:\left(V, B_{V}\right) \rightarrow Y$ satisfies all properties of an lc-trivial fibration, except that the rank of $f_{*} \mathcal{O}_{X}\left(\left\lceil\mathbb{A}^{*}\left(V, B_{V}\right)\right\rceil\right)$ might be bigger than one.
- Both $f:(X, B) \rightarrow Y$ and $h:\left(V, B_{V}\right) \rightarrow Y$ induce the same discriminant and moduli part on $Y$.

The canonical bundle formula for $h:\left(V, B_{V}\right) \rightarrow Y$ is:

$$
\begin{equation*}
K_{V}+B_{V}+(\psi)=h^{*}\left(K_{Y}+B_{Y}+M_{Y}\right) \tag{1.2}
\end{equation*}
$$

Let $E_{V}$ be the sum of all the centers of $\left(V, B_{V}\right)$.
By taking a log resolution of the pairs $(X, B)$ and $\left(V, B_{V}\right)$ and a resolution of $Y$, we can assume that the log smoothness hypotheses of [7, pp. 262-263] and [10, p. 334] are verified (cf. [2, p. 245]). We will refer to this set of properties as the SNC setting.

Lemma 1.2. The following properties hold for the above set-up:
(1) The group $G$ acts naturally on $h_{*} \mathcal{O}_{V}\left(K_{V / Y}+E_{V}\right)$. The eigensheaf corresponding to the eigenvalue $\zeta$ is $\tilde{\mathcal{L}}:=f_{*} \mathcal{O}_{X}(\Gamma-B+E+$ $\left.\left.f^{*} B_{Y}+f^{*} M_{Y}\right\rceil\right)$.
(2) Assume that $h: V \rightarrow Y$ is semi-stable in codimension one. Then $M_{Y}$ is an integral divisor, $\tilde{\mathcal{L}}$ is semi-positive and $\tilde{\mathcal{L}}=O_{Y}\left(M_{Y}\right) \cdot \psi$.

Proof. Since $(\varphi)$ has SNC support, the variety $\tilde{X}$ has canonical singularities and

$$
h_{*} \mathcal{O}_{V}\left(K_{V / Y}+E_{V}\right)=f_{*} \pi_{*}\left(\omega_{\tilde{X} / Y} \otimes \mathcal{O}_{\tilde{X}}(\tilde{E})\right)
$$

The action on $h_{*} \mathcal{O}_{V}\left(K_{V / Y}+E_{V}\right)$ is induced by the one on $\pi_{*}\left(\omega_{\tilde{X} / Y} \otimes \mathcal{O}_{\tilde{X}}(\tilde{E})\right)$, thus the eigensheaf of $\zeta$ is:

$$
\tilde{\mathcal{L}}=f_{*} \mathcal{O}_{X}\left(\left\lceil-B+E+f^{*} B_{Y}+f^{*} M_{Y}\right\rceil\right)
$$

This completes the proof of item (1).
We claim that there exists $Y^{\dagger} \subseteq Y$, an open set such that $\operatorname{codim}(Y \backslash) Y^{\dagger} \geqslant 2$ and $\left.\left(-B_{V}+E_{V}+h^{*} B_{Y}\right)\right|_{h^{-1} Y^{\dagger}}$ is effective and supports no fibers. Indeed, since $h$ is semistable, using the same notation as in [6] or [1, p. 14], there exists $j_{0}$ such that $\gamma_{p}=1-b_{j_{0}}$ (here $w_{j}=1$ for any $j$ ).

Then $1-\gamma_{p}-b_{j_{0}}=0$ and $-B_{V}+h^{*} B_{Y}$ does not contain the fiber over $p$. Since $E_{V}$ is horizontal, the same reasoning holds for $-B_{V}+E_{V}+h^{*} B_{Y}$.

For the effectivity, from formula (1.2) we deduce that the coefficients of $\left(B_{V}\right)^{h}$ are integer, thus they are either 1 or negative. Then $\left(-B_{V}+E_{V}+h^{*} B_{Y}\right)^{h}=\left(-B_{V}+E_{V}\right)^{h}$ is effective. The effectivity of $\left(-B_{V}+E_{V}+h^{*} B_{Y}\right)^{v}=\left(-B_{V}+h^{*} B_{Y}\right)^{v}$ follows from [6], [1, p. 14]. Let $H$ be a general fiber of $h$. By restricting formula (1.2) to $H$, we get:

$$
\left(\left.\psi\right|_{H}\right)+K_{H}+\left.E_{V}\right|_{H}=-\left.\left(B_{V}-E_{V}\right)\right|_{H} \geqslant 0
$$

This implies that there exists an open subset $U \subseteq Y$ such that $\left.\left((\psi)+K_{V / Y}+E_{V}\right)\right|_{U} \geqslant 0$ and $\psi$ is a rational section of $h_{*} \mathcal{O}\left(K_{V / Y}+E_{V}\right)$. Moreover, since by the action of $G$ we have $\psi \mapsto \zeta \psi$, the function $\psi$ is a rational section of $\tilde{\mathcal{L}}$ the eigensheaf of $\zeta$. The sheaf $\tilde{\mathcal{L}}$ has rank one because for general $y \in Y$, we have $\tilde{\mathcal{L}}_{y} \cong H^{0}\left(F,\left.\left\lceil-B+E+f^{*} B_{Y}+f^{*} M_{Y}\right\rceil\right|_{F}\right)=$ $H^{0}\left(F,\left.\lceil-B+E\rceil\right|_{F}\right)$ and the last one is a rank one $\mathbb{C}$-vector space by [2, Definition 2.1(2)]. Thus we can consider $\tilde{\mathcal{L}}$ as a subsheaf of $k(X) \psi$. We prove now that $\left.\tilde{\mathcal{L}}\right|_{Y_{\dagger}}=\left.\mathcal{O}_{Y}\left(M_{Y}\right) \psi\right|_{Y_{\dagger}}$.

Since $\left.\left(-B_{V}+E_{V}+h^{*} B_{Y}\right)\right|_{h^{-1}\left(Y^{\dagger}\right)}$ is effective and $h^{*} M_{Y}-B_{V}+E_{V}+h^{*} B_{Y}=K_{V / Y}+E_{V}$, we have:

$$
\left.\left.h^{*} \mathcal{O}_{V}\left(M_{Y}\right)\right|_{h^{-1}\left(Y^{\dagger}\right)} \subseteq \mathcal{O}_{V}\left(K_{V / Y}+E_{V}\right)\right|_{h^{-1}\left(Y^{\dagger}\right)}
$$

and

$$
\left.\left.h_{*} h^{*} \mathcal{O}_{V}\left(M_{Y}\right)\right|_{Y^{\dagger}} \subseteq h_{*} \mathcal{O}_{V}\left(K_{V / Y}+E_{V}\right)\right|_{Y^{\dagger}}
$$

Now let $a \in k(Y)$ be such that $h^{*} a+K_{V / Y}+E_{V} \geqslant 0$. Since $\left.\left(-B_{V}+E_{V}+h^{*} B_{Y}\right)\right|_{h^{-1}\left(Y^{\dagger}\right)}$ contains no fibers we have $h^{*} a+$ $h^{*} M_{Y} \geqslant 0$, thus $\left.\left.h_{*} \mathcal{O}_{V}\left(K_{V / Y}+E_{V}\right)\right|_{Y^{\dagger}} \subseteq h_{*} h^{*} \mathcal{O}_{V}\left(M_{Y}\right)\right|_{Y^{\dagger}}$. Then $\left.h_{*} \mathcal{O}_{V}\left(K_{V / Y}+E_{V}\right)\right|_{Y^{\dagger}}=\left.h_{*} h^{*} \mathcal{O}_{V}\left(M_{Y}\right)\right|_{Y^{\dagger}}$.

By considering the action of $G$, we obtain the equality between the eigensheaves of $\zeta$. From now on the proof follows exactly the same lines as [2, Lemma 5.2].

Lemma 1.3 (Theorem 4.3 [2]). There exists a finite Galois cover $\tau: Y^{\prime} \rightarrow Y$ from a non-singular variety $Y^{\prime}$ which admits a simple normal crossings divisor supporting $\tau^{-1}\left(\Sigma_{Y}\right)$ and the locus where $\tau$ is not étale, and such that $h^{\prime}: V^{\prime} \rightarrow Y^{\prime}$ is semi-stable in codimension one for some set-up $\left(V^{\prime}, B_{V^{\prime}}\right) \rightarrow\left(X^{\prime}, B_{X^{\prime}}\right) \rightarrow Y^{\prime}$ induced by base change.

The following theorem is a generalization of Theorem 4.4 in [2]. It has been proved in [4] by using variation of mixed Hodge structures. Here we give a proof based on variation of Hodge structures.

Theorem 1.4. Let $f:(X, B) \rightarrow Y$ be an lc-trivial fibration and let $D=\sum_{i=1}^{N} D_{i}$ be the sum of the horizontal lc-centers of $(X, B)$. Assume that:

- we are in the SNC setting;
- the monodromies of $R^{i} f_{0 *} \mathbb{C}_{X_{0} \backslash D_{0}}$ are unipotent $\forall i$ where $Y_{0}=Y \backslash \Sigma_{Y}, X_{0}=f^{-1} Y_{0}, D_{0}=D \cap X_{0}, f_{0}=\left.f\right|_{X_{0} \backslash D_{0}}$.

Let $\rho: Y^{\prime} \rightarrow Y$ be a projective morphism from a non-singular variety $Y^{\prime}$ such that $\rho^{-1} \Sigma_{Y}$ is a simple normal crossings divisor. Let $X^{\prime} \rightarrow\left(X \times Y^{\prime}\right)_{\text {main }}$ be a resolution of the component of $X \times Y^{\prime}$ which dominates $Y^{\prime}$, and let $f^{\prime}: X^{\prime} \rightarrow Y^{\prime}$ be the fiber space induced by the base change. Then for any $i \geqslant 0$ there exists a natural isomorphism $\rho^{*} R^{i} f_{*} \omega_{X / Y}(D) \cong R^{i} f_{*}^{\prime} \omega_{X^{\prime} / Y^{\prime}}\left(D^{\prime}\right)$, where $D^{\prime}$ is the strict transform of $D$, which extends the base change isomorphism over $Y \backslash \Sigma_{Y}$.

First we have to state a preliminary result.

Proposition 1.5 (Theorem 4.4(3) [2]). Let $f: X \rightarrow Y$ be a surjective morphism. Assume that $X$ and $Y$ are smooth and that the locus where $f$ is not smooth is a simple normal crossings divisor $\Sigma_{Y}$. Let $Y_{0}$ be $Y \backslash \Sigma_{Y}$, let $X_{0}$ be $f^{-1} Y_{0}$ and $f=\left.f\right|_{X_{0}}$. Assume that the local systems $R^{i} f_{0 *} \mathbb{C}_{X_{0}}$ have unipotent monodromies around $\Sigma_{Y}$ for any i. Let $\rho: Y^{\prime} \rightarrow Y$ and $X^{\prime}$ be a projective morphism from a non-singular variety $Y^{\prime}$ such that $\rho^{-1} \Sigma_{Y}$ is a simple normal crossings divisor. Let $X^{\prime} \rightarrow\left(X \times Y^{\prime}\right)_{\text {main }}$ be a resolution of the component of $X \times Y^{\prime}$ which dominates $Y^{\prime}$, and let $f^{\prime}: X^{\prime} \rightarrow Y^{\prime}$ be the induced fiber space. Then for any $i \geqslant 0$ there exists a natural isomorphism $\rho^{*} R^{i} f_{*} \omega_{X / Y} \cong R^{i} f_{*}^{\prime} \omega_{X^{\prime} / Y^{\prime}}$.

Proof. Set $\Sigma_{Y^{\prime}}=\rho^{-1} \Sigma_{Y}, Y_{0}^{\prime}=Y^{\prime} \backslash \Sigma_{Y^{\prime}}, X_{0}^{\prime}=f^{\prime-1} Y_{0}^{\prime}$ and $f_{0}^{\prime}=\left.f^{\prime}\right|_{X_{0}^{\prime}}$. The locally free sheaves $H_{0}^{(i)}=R^{m+i} f_{0 *} \mathbb{C}_{X_{0}}$ and $H_{0}^{\prime(i)}=$ $R^{m+i} f_{0 *}^{\prime} \mathbb{C}_{X_{0}^{\prime}}$ are the underlying spaces of variation of Hodge structures of weight $m-i$. In [9, Theorem 2.6, p. 176] is proved that:

$$
\begin{aligned}
& R^{i} f_{*} \omega_{X / Y} \cong \\
& R^{i} f_{*}^{\prime} \omega^{b}\left(R^{m+i} Y^{\prime} f_{*} \mathbb{C}_{X_{0}}\right) \quad \forall i \geqslant 0 \\
& \mathcal{F}^{b}\left(R^{m+i} f_{*}^{\prime} \mathbb{C}_{X_{0}^{\prime}}\right) \quad \forall i \geqslant 0
\end{aligned}
$$

where the right side of the equality denotes the upper canonical extension of the bottom part of the Hodge filtration. Since $H_{0}^{(i)}$ has unipotent local monodromies, the upper canonical extensions coincide with the canonical extensions. Moreover, by the unipotent monodromies assumption, the canonical extension is compatible with base change by [8, Proposition 1, p. 4]. Hence by unicity of the extension the isomorphism $\rho^{*} R^{i} f_{0 *} \omega_{X_{0} / Y_{0}} \cong R^{i} f_{0 *}^{\prime} \omega_{X_{0}^{\prime} / Y_{0}^{\prime}}$ induces an isomorphism $\rho^{*} R^{i} f_{*} \omega_{X / Y} \cong$ $R^{i} f_{*}^{\prime} \omega_{X^{\prime} / Y^{\prime}}$.

Proof of Theorem 1.4. Let $N$ be the number of irreducible components of $D$. We prove the statement by double induction on $N$ and on the dimension $d$ of the fiber.

If $N=0$ or $d=0$ the result follows from Proposition 1.5 . Suppose $N>0$ and consider the exact sequence:

$$
\begin{equation*}
0 \rightarrow \mathcal{O}_{X}(\tilde{D}) \rightarrow \mathcal{O}_{X}(D) \rightarrow \mathcal{O}_{D_{1}}(D) \rightarrow 0 \tag{1.3}
\end{equation*}
$$

where $\tilde{D}=\sum_{i=2}^{N} D_{i}$. Set $\tilde{D}^{\prime}=\sum_{i=2}^{N} D_{i}^{\prime}$ and

$$
\begin{aligned}
& A_{i}=\rho^{*} R^{i} f_{*} \omega_{X / Y}(\tilde{D}), \quad B_{i}=\rho^{*} R^{i} f_{*} \omega_{X / Y}(D), \quad C_{i}=\rho^{*} R^{i} f_{*} \omega_{D_{1} / Y}(\tilde{D}) \\
& A_{i}^{\prime}=R^{i} f_{*}^{\prime} \omega_{X^{\prime} / Y^{\prime}}\left(\tilde{D}^{\prime}\right), \quad B_{i}^{\prime}=R^{i} f_{*}^{\prime} \omega_{X^{\prime} / Y^{\prime}}\left(D^{\prime}\right), \quad C_{i}^{\prime}=R^{i} f_{*}^{\prime} \omega_{D_{1}^{\prime} / Y^{\prime}}\left(\tilde{D}^{\prime}\right) .
\end{aligned}
$$

We have a commutative diagram with exact lines:


The morphisms $\beta$ and $\varepsilon$ are isomorphisms by the inductive hypothesis on $N$. The morphisms $\alpha$ and $\delta$ are isomorphisms by the inductive hypothesis on $d$. Then, by the snake lemma, also $\gamma$ is an isomorphism.

Lemma 1.6. Let $\gamma: Y^{\prime} \rightarrow Y$ be a generically finite projective morphism from a non-singular variety $Y^{\prime}$. Assume there exists a simple normal crossings divisor $\Sigma_{Y^{\prime}}$ on $Y^{\prime}$ which contains $\gamma^{-1} \Sigma_{Y}$, and the locus where $\gamma$ is not étale. Let $M_{Y^{\prime}}$ be the moduli part of the induced set-up $\left(V^{\prime}, B_{V^{\prime}}\right) \rightarrow\left(X^{\prime}, B_{X^{\prime}}\right) \rightarrow Y^{\prime}$. Then $\gamma^{*}\left(M_{Y}\right)=M_{Y^{\prime}}$.

Proof. The proof is exactly the same as that of [2, p. 248]. We just replace $\gamma^{*} h_{*} \mathcal{O}_{V}\left(K_{V / Y}\right)$ with $\gamma^{*} h_{*} \mathcal{O}_{V}\left(K_{V / Y}+E\right)$ and $h_{*}^{\prime} \mathcal{O}_{V^{\prime}}\left(K_{V^{\prime} / Y^{\prime}}\right)$ with $h_{*}^{\prime} \mathcal{O}_{V^{\prime}}\left(K_{V^{\prime} / Y^{\prime}}+E^{\prime}\right)$ and we apply Theorem 1.4 instead of [2, Theorem 4.4].

We now give a sketch of the proof of Theorem 1.1.
Proof of Theorem 1.1. The proof follows the same lines as in [2, p. 249]. We give a sketch here for the reader's convenience. We can suppose that we are in an SNC setting,

$$
\left(V^{\prime}, B_{V}\right) \rightarrow\left(X^{\prime}, B\right) \rightarrow Y^{\prime}
$$

In particular $Y^{\prime}$ is smooth and the divisors $M_{Y^{\prime}}$ and $K_{Y^{\prime}}+B_{Y^{\prime}}$ are $\mathbb{Q}$-Cartier.
Now we prove that $M_{Y^{\prime}}$ is nef. By Lemma 1.3 there exists a finite morphism $\tau: \bar{Y}^{\prime} \rightarrow Y^{\prime}$ such that $\bar{h}^{\prime}: \bar{V}^{\prime} \rightarrow \bar{Y}^{\prime}$ is semistable in codimension one. By Lemma 1.2, the divisor $M_{\bar{Y}^{\prime}}$ is integral and nef. Since $\tau$ is finite we can apply [2, Proposition 5.5] and have $\tau^{*} M_{Y^{\prime}}=M_{\bar{Y}^{\prime}}$. Again, since $\tau$ is finite and $M_{\bar{Y}^{\prime}}$ is nef, also $M_{Y^{\prime}}$ is nef.

Finally, by Lemma 1.6, for any birational morphism $v: Y^{\prime} \rightarrow Y$ we have $v^{*} M_{Y^{\prime}}=M_{Y^{\prime \prime}}$.

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