Mathematical analysis

An extremal problem for polynomials

Un problème extrémal pour les polynômes

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A R T I C L E   I N F O

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A B S T R A C T

We give a solution to an extremal problem for polynomials, which asks for complex numbers \( \alpha_0, \ldots, \alpha_n \) of unit magnitude that minimise the largest supremum norm on the unit circle for all polynomials of degree \( n \) whose \( k \)-th coefficient is either \( \alpha_k \) or \( -\alpha_k \).

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R É S U M É

Nous donnons dans ce papier une solution à un problème extrémal sur les polynômes, qui est de trouver des nombres complexes \( \alpha_0, \ldots, \alpha_n \) de module égal à 1 qui minimisent, sur le cercle unité, la plus grande borne supérieure de la norme pour tous les polynômes de degré \( n \) qui ont pour \( k \)- coefficient \( \alpha_k \) ou \( -\alpha_k \).

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1. Results

Extremal problems for polynomials are typically of the following form. Let \( f(z) \) be a polynomial with coefficients restricted to be in a subset of the complex numbers (often, this set is \([-1, 1]\)). How well can \(|f(z)|\) approximate a constant function when \( z \) ranges over the unit circle? This metaproblem has many variations, most of which are open (see, for example, Littlewood [8], Borwein [1], and Erdélyi [2] for surveys on selected problems). To quantify the gap between \(|f(z)|\) and a constant function, different norms on the unit circle have been considered. The supremum norm:

\[
\|f\| = \max_{|z|=1} |f(z)|
\]

has received particular attention. For example, Erdös [3, Problem 22], [4], and Littlewood [7] were interested in the minimum of \( \|f_n\| \), where \( f_n \) is a polynomial of degree \( n-1 \) with coefficients of absolute value 1. In particular, Erdös [4] conjectured that there is some \( c > 0 \) such that \( \|f_n\|/\sqrt{n} \geq 1 + c \) for all polynomials \( f_n \) of degree \( n-1 \) (for every \( n \geq 1 \)), whose coefficients have absolute value 1. Kahane [5] proved that there is no such \( c \), but the modified conjecture where \( f_n \) is restricted to have coefficients 1 or \(-1\) only remains open.

In this paper, we study an extremal problem in which the goal is to minimise the largest supremum norm for polynomials whose \( k \)-th coefficient is either \( \alpha_k \) or \( -\alpha_k \), where the minimisation is over the complex numbers \( \alpha_0, \alpha_1, \ldots \) satisfying \( |\alpha_0| = |\alpha_1| = \cdots = 1 \). Specifically, defining:

\[
B_n(\phi_0, \ldots, \phi_{n-1}) = \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in \{-1, 1\}} \left\| \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \phi_k} z^k \right\|
\]

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for \(\phi_0, \ldots, \phi_{n-1} \in [0, 1]\), we are interested in the minimum:

\[
b(n) = \min_{\phi_0, \ldots, \phi_{n-1} \in [0, 1]} B_n(\phi_0, \ldots, \phi_{n-1}).
\]

This minimisation problem is also related to an optimisation problem in communications engineering and, in this context, variations of the problem have been studied by Tarokh and Jafarkhani [10], Litsyn and Wunder [6], and Schmidt [9], among others.

It is easy to see that \(b(n) \leq n\) and that \(b(1) = 1\) and \(b(2) = 2\), but the value of \(b(n)\) is unknown for all \(n \geq 3\). One might be tempted to conjecture that \(b(n)\) is monotonically increasing and \(b(n)/n\) is monotonically decreasing, but this also remains unknown.

Our main result is the following.

**Theorem 1.** We have:

\[
\lim_{n \to \infty} \frac{b(n)}{n} = \frac{2}{\pi}.
\]

We prove Theorem 1 in Propositions 2 and 3 below. Proposition 2 establishes that \(b(n)/n > 2/\pi\) for all \(n \geq 1\) and Proposition 3 gives explicit constructions of \(\phi_0, \ldots, \phi_{n-1}\) for which \(B_n(\phi_0, \ldots, \phi_{n-1})/n\) approaches \(2/\pi\) arbitrarily closely.

**Proposition 2.** For each \(n \geq 1\), we have:

\[
\frac{b(n)}{n} > \frac{2}{\pi}.
\]

**Proof.** We have:

\[
B_n(\phi_0, \ldots, \phi_{n-1}) \geq \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in [-1, 1]} \left| \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \phi_k} \right| = \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in [-1, 1]} \max_{\psi \in [0, 1]} \Re \left( \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \phi_k} e^{2\pi i \psi} \right)
\]

\[
= \max_{\psi \in [0, 1]} \left| \sum_{k=0}^{n-1} \cos(2\pi (\phi_k + \psi)) \right|.
\]

Since the sum cannot be constant for all \(\psi\), we can further bound this expression as follows:

\[
B_n(\phi_0, \ldots, \phi_{n-1}) > \int_0^1 \left| \sum_{k=0}^{n-1} \cos(2\pi (\phi_k + \psi)) \right| d\psi
\]

\[
= \sum_{k=0}^{n-1} \int_0^1 \left| \cos(2\pi \psi) \right| d\psi = \frac{2n}{\pi},
\]

as required. \(\square\)

**Proposition 3.** Let \(\alpha\) be an irrational number and write \(\phi_k = \alpha k^2\). Then

\[
\lim_{n \to \infty} \frac{B_n(\phi_0, \ldots, \phi_{n-1})}{n} = \frac{2}{\pi}.
\]

**Proof.** We have:

\[
B_n(\phi_0, \ldots, \phi_{n-1}) = \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in [-1, 1]} \max_{\theta \in [0, 1]} \left| \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \alpha k^2} e^{2\pi i \theta k} \right|
\]

\[
= \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in [-1, 1]} \max_{\theta, \psi \in [0, 1]} \Re \left( \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i (\alpha k^2 + \theta k + \psi)} \right)
\]

\[
= \max_{\theta, \psi \in [0, 1]} \left| \sum_{k=0}^{n-1} \cos(2\pi (\alpha k^2 + \theta k + \psi)) \right|.
\]
Since $\alpha$ is irrational, we obtain the following consequence of a celebrated result on equi-distributed sequences modulo 1 due to Weyl [11, Satz 9]:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i m(\alpha k^2 + \theta k + \psi)} = 0 \quad \text{for every integer } m \neq 0.$$ 

Moreover, the convergence is uniform for all $\theta, \psi \in \mathbb{R}$. It then follows easily [11, pp. 314–315] that:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\cos(2\pi (\alpha k^2 + \theta k + \psi))| = \int_0^1 |\cos 2\pi x| \, dx$$

uniformly for all $\theta, \psi \in \mathbb{R}$. Hence, from (1),

$$\lim_{n \to \infty} \frac{B_n(\phi_0, \ldots, \phi_{n-1})}{n} = \lim_{n \to \infty} \max_{\theta, \psi \in [0,1]} \frac{1}{n} \sum_{k=0}^{n-1} |\cos(2\pi (\alpha k^2 + \theta k + \psi))| = \max_{\theta, \psi \in [0,1]} \int_0^1 |\cos 2\pi x| \, dx = \frac{2}{\pi},$$

as required. $\square$

References


[7] J.E. Littlewood, On polynomials $\sum_{n=0}^{N} \pm e^{i\theta n}, \sum_{n=0}^{N} e^{i\theta n}, \sum_{n=0}^{N} e^{i\theta n}, \sum_{n=0}^{N} e^{i\theta n}$, J. Lond. Math. Soc. 41 (1966) 367–376.


