



Probability theory

Moment formulae for general point processes

*Formules de moments pour des processus ponctuels quelconques*

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ABSTRACT

The goal of this paper is to generalize most of the moment formulae obtained in [6]. More precisely, we consider a general point process μ , and show that the relevant quantities to our problem are the so-called Papangelou intensities. Then, we show some general formulae to recover the moment of order n of the stochastic integral of a random process. We will use these extended results to study a random transformation of the point process. The full proofs can be found in [2].

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RÉSUMÉ

L'objectif de ce papier est de généraliser la plupart des formules de moments obtenues dans [6]. Nous calculons les moments de tous ordres des intégrales stochastiques d'un processus ponctuel en fonction de son intensité de Papangelou. Nous utilisons ensuite ces résultats pour généraliser la formule d'isométrie de Skorohod pour les intégrales stochastiques compensées. Enfin, nous étudions la loi d'une transformation aléatoire du processus ponctuel sous une condition de cyclicité qui généralise la notion d'adaptabilité à un espace de dimension quelconque.

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Les processus ponctuels constituent un cadre très général permettant de modéliser un nombre important de phénomènes. Ainsi, un nombre important d'outils ont été développés pour permettre une étude fine d'un processus ponctuel quelconque (cf. [1] et [3]). Pourtant, en pratique, c'est souvent le processus ponctuel de Poisson qui est favorisé pour nombre de ses propriétés abondamment étudiées. Notre objectif est de généraliser les formules de [4,5] au cas d'un processus ponctuel quelconque, ou presque. Pour obtenir nos résultats, nous ne pouvons pas recourir aux mêmes types de preuves que dans [6], vu que celles-ci reposent principalement sur des outils de calcul de Malliavin développés uniquement dans le cadre poissonnien. Pour être plus précis, nous cherchons à calculer les moments de tous ordres d'objets du type $\sum_{x \in \xi} u(x, \xi)$, où ξ est une configuration sur un espace polonais et u un processus stochastique. Ainsi, nous obtenons le Théorème 1, qui généralise ce qui a été obtenu dans [6] pour un processus ponctuel de Poisson. Ce théorème généralise aussi ceux de [1] qui concernent un processus ponctuel général, mais uniquement pour une fonction $u(x, \xi)$ ne dépendant que de x . Notre démonstration est basée principalement sur l'équation de Georgii–Nguyen–Zessin rappelée en (2). Elle implique la notion

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d'intensité de Papangelou $c(x, \xi)$, définie intuitivement comme la densité de probabilité d'avoir une particule en x sachant que l'on observe la configuration ξ .

Puis nous en déduisons un opérateur de divergence, traditionnellement défini pour un processus ponctuel de Poisson, et que nous généralisons à un processus ponctuel quelconque via la formule (4). Nous sommes de même en mesure de calculer les moments de $\delta(u)$, mais surtout en déduisons le Corollaire 4, qui généralise la formule d'isométrie bien connue dans le cas du processus ponctuel de Poisson. Enfin, nous considérons une transformation $\tau(x, \xi)$ qui perturbe les particules x d'une configuration selon $\tau(\xi) := \sum_{x \in \xi} \delta_{\tau(x, \xi)}$, où ici δ est la fonction de Dirac. Nous caractérisons dans le Théorème 5 la mesure transformée. Les conditions imposées ici sur la transformation τ sont les mêmes que dans le cas du processus ponctuel de Poisson. Les preuves de tous nos résultats peuvent être trouvées dans [2].

1. Introduction

Point processes constitute a general framework used to model a wide variety of phenomena. The underlying theory is well understood, and the relevant literature is abundant (see [1] and [4] for example). However, we have a much deeper understanding of the Poisson point process, which is one of the reasons for its use in a lot of practical cases. In particular, one has a chaos-expansion of Poisson functionals, concentration inequalities, moment formulae, etc. [5,6]. On the other hand, one lacks most of these tools for more general point processes. Our goal is to obtain moment formulae for very general point processes that are only required to have a Papangelou intensity.

We will follow the path of [7], in which some moment formulae were obtained for the Poisson point process (PPP). In particular, the main result of [7] is the following formula, obtained in the case of the PPP:

$$\mathbb{E}\left[\left(\int u_y(\xi)\xi(dy)\right)^n\right] = \sum_{\{P_1, \dots, P_k\} \in \mathcal{T}_n} \mathbb{E}\left[\int_E u_{x_1}^{|P_1|} \dots u_{x_k}^{|P_k|} (\xi \cup x_1 \cup \dots \cup x_k) \lambda(dx_1) \dots \lambda(dx_k)\right], \quad (1)$$

where \mathcal{T}_n is the set of all partitions of $\{1, \dots, n\}$, $|P_i|$ is the cardinality of P_i , $i = 1, \dots, k$, and $u : E \times \mathcal{X} \rightarrow \mathbb{R}$ is a nonnegative measurable process.

The proofs in [7] are mostly based on Malliavin calculus based formulas. Our proofs, which can be found in [2], valid for almost any point processes, are mainly based on the Georgii–Nguyen–Zessin formula. Our results also allow us to study random transformations of point processes. In the case of a general point process μ , we will consider a random transformation τ , such that each particle x of the configuration ξ is moved to $\tau(x, \xi)$. Then, we obtain an explicit characterization of $\tau\mu$ if we only assume that τ is invertible and satisfies a cyclicity condition.

2. Notations and general results

Let E be a Polish space, $\mathcal{O}(E)$ the family of all non-empty open subsets of E and \mathcal{B} denote the corresponding Borel σ -algebra. λ is a Radon measure on (E, \mathcal{B}) . Let \mathcal{X} (resp. \mathcal{X}_0) be the space of locally finite (resp. finite) subsets in E , also called the configuration space, equipped with the vague topology. We denote by \mathcal{F} (resp. \mathcal{F}_0) the corresponding σ -algebra. For a given $\xi = \sum_{y \in \xi} \varepsilon_y$, we will usually view ξ as a set, and write $\xi \cup y_0 = \xi \cup \{y_0\}$ for the addition of a particle at y_0 and $\xi \setminus y_0 = \xi \setminus \{y_0\}$ for the removal of a particle at y_0 . Moreover, for a measurable nonnegative $u : E \times \mathcal{X} \rightarrow \mathbb{R}$, we will often use the notation $\int u(y, \xi), \xi(dy) := \sum_{y \in \xi} u(y, \xi)$. A point process μ is said to have correlation functions $(\rho_n)_{n \in \mathbb{N}}$ if for any A_1, \dots, A_n disjoint bounded Borel subsets of E ,

$$\mathbb{E}\left[\prod_{i=1}^n \xi(A_i)\right] = \int_{A_1 \times \dots \times A_n} \rho_n(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n).$$

Recall that ρ_1 is the mean density of particles with respect to λ , and $\rho_n(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n)$ is the probability of finding a particle in the vicinity of each x_i , $i = 1, \dots, n$.

The reduced Campbell measure of a point process μ is the measure C_μ on the product space $(E \times \mathcal{X}, \mathcal{B} \otimes \mathcal{F})$ defined by

$$C_\mu(A \times B) = \int \sum_{x \in \xi} \mathbf{1}_A(x) \mathbf{1}_B(\xi \setminus x) \mu(d\xi),$$

where $A \in \mathcal{B}$ and $B \in \mathcal{F}$. We will thus assume throughout this paper that the following condition is fulfilled:

Hypothesis 1 (*Condition* (Σ_λ)). The point process μ is assumed to satisfy condition (Σ_λ) , i.e. $C_\mu \ll \lambda \otimes \mu$.

Henceforth, any Radon–Nikodym density c of C_μ relative to $\lambda \otimes \mu$ is called a version of the Papangelou intensity of μ . More generally, we set

$$c(\{x_1, \dots, x_n\}, \xi) = \prod_{k=1}^n c(x_k, \xi \cup x_1 \cup \dots \cup x_{k-1}),$$

where we have used the convention $x_0 := \emptyset$. **Hypothesis 1**, along with the definition of the reduced Campbell measure, allows us to write the following important identity, known as the Georgii–Nguyen–Zessin identity:

$$\int_{\mathcal{X}} \sum_{y \in \xi} u(y, \xi \setminus y) \mu(dy) = \int_{\mathcal{X}} \int_E u(z, \xi) c(z, \xi) \lambda(dz) \mu(d\xi), \quad (2)$$

for all C_μ -measurable nonnegative functions $u : E \times \mathcal{X} \rightarrow \mathbb{R}$.

3. Moment formulae

The main theorem, which is mainly based on the Georgii–Nguyen–Zessin formula, is the following:

Theorem 1. For any $n \in \mathbb{N}$, any measurable nonnegative functions $u_k : E \times \mathcal{X} \rightarrow \mathbb{R}$, $k = 1, \dots, n$, and any bounded function F on \mathcal{X} , we have

$$\mathbb{E} \left[F \prod_{k=1}^n \int u_k(y, \xi) \xi(dy) \right] = \sum_{k=1}^n \sum_{\mathcal{P} \in \mathcal{T}_n^k} \mathbb{E} \left[\int_{E^k} F(\xi \cup x) u^{\mathcal{P}}(x, \xi \cup x) c(x, \xi) \lambda_k(dx) \right],$$

where \mathcal{T}_n^k is the set of all partitions of $\{1, \dots, n\}$ into k subsets. Here, for $\mathcal{P} = \{P_1, \dots, P_k\} \in \mathcal{T}_n^k$, we have used the compact notation $x := (x_1, \dots, x_k)$, as well as $\lambda_k(dx) := \lambda(dx_1) \dots \lambda(dx_k)$ and

$$u^{\mathcal{P}}(x, \xi) := \prod_{l=1}^k \prod_{i \in P_l} u_i(x_l, \xi).$$

We can then establish the formula analogous to (1) for general point processes.

Corollary 3.1. For any $n \in \mathbb{N}$, and any measurable nonnegative function v on E , we have

$$\begin{aligned} & \mathbb{E} \left[F \left(\int v(y) \xi(dy) \right)^n \right] \\ &= \sum_{k=1}^n \sum_{\{P_1, \dots, P_k\} \in \mathcal{T}_n^k} \int_{E^k} v(x_1)^{|P_1|} \dots v(x_k)^{|P_k|} \mathbb{E} [F(\xi \cup x_1 \cup \dots \cup x_k) c(\{x_1, \dots, x_k\}, \xi)] \lambda(dx_1) \dots \lambda(dx_k). \end{aligned}$$

In order to go further, we introduce another hypothesis.

Hypothesis 2. We assume that for any integer i and any positive integer k ,

$$\mathbb{E} \left[\xi(\Lambda)^k \left(\int c(z, \xi) \lambda(dz) \right)^i \right] < \infty,$$

for all compact sets $\Lambda \subset E$.

This hypothesis is for instance satisfied for processes with repulsion or attraction like determinantal and permanantal point processes (see [3,8]). Then, we prove the following:

Proposition 2. Assume that **Hypothesis 2** is verified. Then, for any $n \in \mathbb{N}$, any bounded process $u : E \times \mathcal{X} \rightarrow \mathbb{R}$ with compact support on E , and any bounded function F on \mathcal{X} , we have

$$\begin{aligned} \mathbb{E} \left[F \left(\int u(y, \xi) v(dy) \right)^n \right] &= \sum_{i=0}^n (-1)^i \binom{n}{i} \sum_{k=1}^{n-i} \sum_{\{P_1, \dots, P_k\} \in \mathcal{T}_{n-i}^k} \mathbb{E} \left[\int_{E^k} c(x, \xi) F(\xi \cup x) \prod_{l=1}^k u(x_l, \xi \cup x)^{|P_l|} \right. \\ &\quad \times \left. \left(\int u(z, \xi \cup x) c(z, \xi \cup x) \lambda(dz) \right)^i \lambda_k(dx) \right]. \end{aligned} \quad (3)$$

4. Generalized divergence

In the context of Malliavin calculus for Poisson process, it is well known that the difference operator admits the compensated stochastic integral as an adjoint. The previous formulas imply an elegant integration by parts formula for general point processes, or equivalently give the characterization of the adjoint of the difference operator.

Definition 4.1 (*Difference operator*). For $F : \mathcal{X} \rightarrow \mathbb{R}$, we define DF , the difference operator applied to F , as follows:

$$\begin{aligned} DF : E \times \mathcal{X} &\longrightarrow \mathbb{R}, \\ (x, \xi) &\longmapsto D_x F(\xi) = F(\xi \cup x) - F(\xi \setminus x). \end{aligned}$$

Definition 4.2 (*Divergence operator*). We say that a measurable $u : E \times \mathcal{X} \rightarrow \mathbb{R}$ belongs to $\text{Dom}(\delta)$ whenever $\mathbb{E}[\int |u(y, \xi)| \times c(y, \xi) \lambda(dy)]$ is finite. Then, for $u \in \text{Dom}(\delta)$, we define $\delta(u)$ as

$$\delta(u) = \int u(y, \xi \setminus y) \xi(dy) - \int u(y, \xi) c(y, \xi) \lambda(dy). \quad (4)$$

It follows from [Proposition 2](#) that we have the following formula.

Corollary 3 (*Duality relation*). For any bounded function F on \mathcal{X} , and a process $u : (x, \xi) \mapsto u(x, \xi)$, C_μ -measurable, and in $\text{Dom}(\delta)$, we have

$$\mathbb{E}[F\delta(u)] = \mathbb{E}\left[\int_E D_z F(\xi) u(z, \xi) c(z, \xi) \lambda(dz)\right]. \quad (5)$$

As a corollary, we obtain a modified isometry formula.

Corollary 4. For a process $u : (x, \xi) \mapsto u(x, \xi)$, C_μ -measurable, and in $\text{Dom}(\delta)$, we have

$$\begin{aligned} \mathbb{E}[\delta(u)^2] &= \mathbb{E}\left[\int u(y, \omega)^2 c(y, \omega) d\lambda(y)\right] + \mathbb{E}\left[\iint D_y u(z, \omega) D_z u(y, \omega) c(\{y, z\}, \omega) d\lambda(y) d\lambda(z)\right] \\ &\quad - \mathbb{E}\left[\iint u(z, \omega) u(y, \omega) (c(\{y, z\}, \omega) - c(y, \omega) c(z, \omega)) d\lambda(y) d\lambda(z)\right]. \end{aligned}$$

It is worthwhile to note that if the first two terms do appear in the corresponding formula for Poisson point processes, the last term is due to the structure of the correlations.

5. Random transformation of the point process

The goal of this section is to give an application of the previous moment formulae. As was explained in the introduction, we wish to study a random transformation of the point process measure μ . Hereafter, we consider the following condition.

Hypothesis 3 (*Cyclicity condition*). For $u : E \times \mathcal{X} \rightarrow \mathbb{R}$, we assume that u satisfies

$$D_{x_1} \dots D_{x_k} u(x_1, \xi) \dots u(x_k, \xi) = 0, \quad \text{for } C_\mu\text{-a.e. } (x, \xi) \in E \times \mathcal{X}$$

for all $k \geq 1$.

Now, let us consider a random shifting $\tau : E \times \mathcal{X} \rightarrow E$. For $\xi \in \mathcal{X}$, consider the image measure of ξ by τ , which we will call the random transformation $\tau_*(\xi)$, defined as

$$\tau_*(\xi) = \sum_{x \in \xi} \delta_{\tau(x, \xi)},$$

and thus τ_* shifts each point of the configuration in the direction τ . We wish to study the effect of the transformation on the underlying measure μ under sufficiently strong conditions on τ . The following hypotheses will be considered:

- (H1) The random transformation τ_* satisfies [Hypothesis 3](#), in the sense that for any $u : \mathcal{X} \rightarrow \mathbb{R}$, $u \circ \tau_*$ verifies [Hypothesis 3](#).
- (H2) For a.e. $\xi \in \mathcal{X}$, $\tau(\cdot, \xi)$ is invertible, and we will note its inverse $\tau^{-1}(x, \xi)$, $x \in E$, $\xi \in \mathcal{X}$. We will also denote by $\tau_*^{-1}(\xi)$ the image measure of ξ by τ^{-1} .

We are now in a position to state the main theorem of this section.

Theorem 5. Let $\tau : E \times \mathcal{X} \rightarrow E$ be a random shifting as defined previously, and satisfying (H1) and (H2). Let us assume that τ maps λ to σ , i.e. $\tau(\cdot, \xi)\lambda = \sigma$, $\xi \in \mathcal{X}$, where σ is a fixed measure on (E, \mathcal{B}) . Then, $\tau_*\mu$ has correlation functions with respect to σ that are given by

$$\rho_\tau(x_1, \dots, x_k) = \mathbb{E}[c(\{\tau^{-1}(x_1, \xi), \dots, \tau^{-1}(x_k, \xi)\}, \xi)], \quad x_1, \dots, x_k \in E. \quad (6)$$

This theorem directly generalizes all known results [6]. Indeed, consider the following corollary:

Corollary 6. Let $\mu = \pi^{d\lambda}$ be the Poisson measure with intensity λ . Let $\tau : E \times \mathcal{X} \rightarrow E$ be a random transformation satisfying (H1) and (H2). Let us assume that τ maps λ to σ , i.e. $\tau(\cdot, \xi)\lambda = \sigma$, $\xi \in \mathcal{X}$. Then, τ maps $\pi^{d\lambda}$ to $\pi^{d\sigma}$.

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