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Two results on φ -normal functions $\stackrel{\diamond}{\Rightarrow}$

Deux résultats sur les fonctions φ -normales

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ABSTRACT

In this paper, we obtain two results on φ -normal functions, which extend some related results due to Lappan, and Aulaskari–Rättyä. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Dans cette note, nous obtenons deux résultats sur les fonctions φ -normales, qui étendent des résultats connexes dus à Lappan et Aulaskari–Rättyä. © 2013 Académie des sciences, Published by Elsevier Masson SAS, All rights reserved.

1. Introduction

Let $\Delta = \{z: |z| < 1\}$ be the unit disc in the complex plane \mathbb{C} , and let $\mathcal{M}(\Delta)$ denote the set of all meromorphic functions in Δ . A function $f \in \mathcal{M}(\Delta)$ is called a normal function, in the sense of Lehto and Virtanen [6], if

$$\sup_{z\in\Delta} (1-|z|^2) f^{\#}(z) < \infty,$$

where $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ is the spherical derivative of f. An increasing function $\varphi : [0, 1) \to (0, \infty)$ is called smoothly increasing if

$$\varphi(r)(1-r) \to \infty$$
, as $r \to 1^-$

and

$$R_a(z) = \frac{\varphi(|a| + z/\varphi(|a|))}{\varphi(|a|)} \to 1, \quad \text{as } |a| \to 1^-$$

uniformly on compact subsets of \mathbb{C} . For a given such φ , we call a function $f \in \mathcal{M}(\Delta)$ is φ -normal (see [1,2]) if

$$\sup_{z\in\Delta}\frac{f^{\#}(z)}{\varphi(|z|)}<\infty.$$

Let \mathcal{N}^{φ} denote the class of all φ -normal functions, and let \mathcal{N} denote the class of all normal functions. Clearly, $\mathcal{N} \subset \mathcal{N}^{\varphi}$.

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For a positive integer k, the expression $|f^{(k)}(z)|/(1+|f(z)|^{k+1})$ can be viewed as an extension of the spherical derivative of f, which is introduced by Lappan [5]. In [5], Lappan also proved

Theorem A. Let $f \in \mathcal{M}(\Delta)$. If $f \in \mathcal{N}$, then for each positive integer k,

$$\sup_{z\in\Delta} (1-|z|^2)^k \frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}} < \infty.$$

The well-known Lappan five-point theorem [4] says that if $\sup\{(1 - |z|^2)f^{\#}(z): z \in \Delta \cap f^{-1}(E)\}$ is bounded for some five-point *E* subset of the extended plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then $f \in \mathcal{N}$. Recently, R. Aulaskari and J. Rättyä [2] got a version of Lappan five-point theorem for φ -normal functions, as follows.

Theorem B. Let $\varphi : [0, 1) \to (0, \infty)$ be smoothly increasing, k be a positive integer, and let $f \in \mathcal{M}(\Delta)$. If there exists a set E of five distinct points in $\hat{\mathbb{C}}$ such that:

$$\sup_{z\in\Delta\cap f^{-1}(E)}\frac{f^{\#}(z)}{\varphi(|z|)}<\infty,$$

then $f \in \mathcal{N}^{\varphi}$.

In this paper, we prove the following results.

Theorem 1. Let $\varphi : [0, 1) \to (0, \infty)$ be smoothly increasing and $f \in \mathcal{M}(\Delta)$. If $f \in \mathcal{N}^{\varphi}$, then for each positive integer k,

$$\sup_{z \in \Delta} \frac{1}{\varphi(|z|)^k} \frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} < \infty$$

Theorem 2. Let $\varphi : [0, 1) \to (0, \infty)$ be smoothly increasing, k be a positive integer, and let $f \in \mathcal{M}(\Delta)$, and suppose that there exists M > 0 such that $\max_{0 \le i \le k-1} |f^{(i)}(z)| \le M$ whenever f(z) = 0 and $z \in \Delta$. If there exists a set E of k + 4 distinct points in $\hat{\mathbb{C}}$ such that:

$$\sup_{z\in\Delta\cap f^{-1}(E)}\frac{1}{\varphi(|z|)^k}\frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}}<\infty,$$

then $f \in \mathcal{N}^{\varphi}$.

Remark. Clearly, Theorem 1 extends Theorem A, and our method to prove Theorem 1 is different from that in [5]. The condition "max $_{0 \le i \le k-1} |f^{(i)}(z)| \le M$ whenever f(z) = 0" in Theorem 2 holds naturally for k = 1. So Theorem 2 is an extension of Lappan five-point theorem and Theorem B.

2. Lemmas

Let *f* be a nonconstant meromorphic function in \mathbb{C} . We shall use the following standard notations of value distribution theory (see [3,8]):

 $T(r, f), m(r, f), N(r, f), \bar{N}(r, f), \dots$

We denote by S(r, f) any function satisfying $S(r, f) = o\{T(r, f)\}, r \to \infty$, possibly outside a set with finite measure. We use $\bar{N}_{(2}(r, f))$ to denote the Nevanlinna counting function of the poles of f with multiplicity ≥ 2 .

Lemma 1. (See [3,8].) Let f be a nonconstant meromorphic function in \mathbb{C} , and let a_1, a_2, \ldots, a_q $(q \ge 3) \in \mathbb{C} \cup \{\infty\}$ be distinct complex numbers, and $k \in \mathbb{N}$. Then

(1) $(q-2)T(r, f) \leq \sum_{i=1}^{q} \bar{N}(r, \frac{1}{f-a_i}) + S(r, f).$ (2) $T(r, f^{(k)}) \leq (k+1)T(r, f) + S(r, f).$

The next lemma reveals a close relationship between φ -normal functions and normal families, which is a direct consequence of Marty's theorem; it can be founded in [1,2].

Lemma 2. Let $\varphi : [0, 1) \to (0, \infty)$ be smoothly increasing, and let $f \in \mathcal{M}(\Delta)$. Then $f \in \mathcal{N}^{\varphi}$ if and only if the family $\{f(a + z/\varphi(|a|)): a \in \Delta\}$ is a normal in Δ .

The following is a version of Lohwater–Pommerenke theorem [7] for \mathcal{N}^{φ} (see [1,2]).

Lemma 3. Let $\varphi : [0, 1) \to (0, \infty)$ be smoothly increasing, and $f \in \mathcal{M}(\Delta)$. If $f \notin \mathcal{N}^{\varphi}$, then there exist a sequence of points $z_n \in D$, two sequences of positive numbers ρ_n , σ_n with $\sigma_n \to 0$, and a constant c > 0 satisfying $\varphi(|z_n|)\rho_n \leq c\sigma_n$ such that $f(z_n + \rho_n\zeta)$ spherically and uniformly converges to a nonconstant meromorphic function on each compact subset of \mathbb{C} .

3. Proof of theorems

Proof of Theorem 1. Theorem 1 is true for k = 1 by the definition of the φ -normal function. Suppose that Theorem 1 is not true for $k \ge 2$, then there exists a sequence $\{z_n\} \subset \Delta$ such that:

$$\frac{1}{\varphi(|z_n|)^k} \frac{|f^{(k)}(z_n)|}{1+|f(z_n)|^{k+1}} \to \infty, \quad n \to \infty.$$

$$\tag{1}$$

Set the family:

$$\mathcal{G} = \left\{ g_n(z) = f\left(z_n + z/\varphi(|z_n|) \right) \right\}$$

By Lemma 2, \mathcal{G} is a normal family in Δ . Then, for each sequence $\{g_n\} \in \mathcal{G}$, there exists a subsequence of $\{g_n\}$ (without loss of generality, we still denote by $\{g_n\}$ for convenience) such that $g_n(z) \to g(z)$ converges spherically locally uniformly in Δ , where g(z) is a meromorphic function (possibly infinity identically).

We distinguish two cases.

Case 1. $g(z) \equiv \infty$. Then $1/g_n \to 0$ in Δ , and thus $(1/g_n)^{(i)} \to 0$ for positive integer *i*. In particular, $g'_n/g_n^2 = -(1/g_n)' \to 0$. On the other hand, an elementary calculation yields:

$$\frac{g_n^{(k)}}{g_n^{k+1}} = -\frac{1}{g_n^{k-1}} \left(\frac{1}{g_n}\right)^{(k)} + P\left(\frac{g_n'}{g_n^2}, \frac{g_n''}{g_n^3}, \dots, \frac{g_n^{(k-1)}}{g_n^k}\right),$$

where $P(w_1, w_2, ..., w_{k-1})$ is a polynomial in $w_1, w_2, ..., w_{k-1}$ with integer coefficients. By induction, we have $\frac{g_n^{(k)}(z)}{g_n^{k+1}(z)} \to 0$ in Δ . It follows that:

$$\frac{|g_n^{(k)}(z)|}{1+|g_n(z)|^{k+1}} \leqslant \left|\frac{g_n^{(k)}(z)}{g_n^{k+1}(z)}\right| \to 0$$
(2)

in Δ . Note that:

.....

$$\frac{|g_n^{(k)}(0)|}{1+|g_n(0)|^{k+1}} = \frac{1}{\varphi(|z_n|)^k} \frac{|f^{(k)}(z_n)|}{1+|f(z_n)|^{k+1}}.$$
(3)

Then, from (2) and (3), we have:

$$\frac{1}{\varphi(|z_n|)^k} \frac{|f^{(k)}(z_n)|}{1+|f(z_n)|^{k+1}} \to 0, \quad \text{as } n \to \infty,$$

which contradicts (1).

Case 2. $g(z) \neq \infty$. Now we divide into subcases.

Case 2.1. $g(0) \neq \infty$.

Then there exists $0 < \delta < 1$ such that g(z) is holomorphic in $\Delta_{\delta} = \{z: |z| < \delta\}$, and hence $g_n(z)$ -for n sufficiently large-are holomorphic. Since $g_n(z) \rightarrow g(z)$, we get:

$$\frac{|g_n^{(k)}(z)|}{1+|g_n(z)|^{k+1}} \to \frac{|g^{(k)}(z)|}{1+|g(z)|^{k+1}}, \quad z \in \Delta_{\delta}.$$

Letting $M_1 = |g^{(k)}(0)|/(1 + |g(0)|^{k+1})$, then for sufficiently large *n*:

$$\frac{|g_n^{(k)}(0)|}{1+|g_n(0)|^{k+1}} \le M_1+1.$$

This and (3) give:

$$\frac{1}{\varphi(|z_n|)^k} \frac{|f^{(k)}(z_n)|}{1+|f(z_n)|^{k+1}} \leqslant M_1+1,$$

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which is a contradiction.

Case 2.2. $g(0) = \infty$.

Then we can find $0 < \delta < 1$ such that g(z) is holomorphic and $|g(z)| \ge 2$ in $\Delta'_{\delta} = \{z: 0 < |z| < \delta\}$, and thus $g_n(z)$ is holomorphic and $|g_n(z)| \ge 2$ in Δ'_{δ} for sufficiently large *n*. So we know that:

$$\frac{g^{(k)}(z)}{1+g^{k+1}(z)} \quad \text{and} \quad \frac{g^{(k)}_n(z)}{1+g^{k+1}_n(z)} \quad (\text{for sufficiently large } n)$$

are holomorphic in Δ_{δ} , and

$$\frac{g_n^{(k)}(z)}{1+g_n^{k+1}(z)} \to \frac{g^{(k)}(z)}{1+g^{k+1}(z)}, \quad z \in \Delta'_{\delta}.$$
(4)

The maximum modulus principle implies that (4) still holds in Δ_{δ} . Letting $M_2 = |g^{(k)}(0)|/|1 + |g(0)|^{k+1}|$, for sufficiently large *n*, we have:

$$\frac{|g_n^{(k)}(0)|}{1+|g_n(0)|^{k+1}} \leqslant \left|\frac{g_n^{(k)}(0)}{1+g_n(0)^{k+1}}\right| \leqslant M_2+1.$$

As in Case 2.1, we arrive at a contradiction. This completes the proof of Theorem 1. \Box

Proof of Theorem 2. Suppose that $f \notin \mathcal{N}^{\varphi}$. Then, by Lemma 3, there exist points $z_n \in \Delta$ tending to the boundary, positive numbers ρ_n with $\varphi(|z_n|)\rho_n \to 0$ such that:

$$g_n(\zeta) = f(z_n + \rho_n \zeta) \to g(\zeta) \tag{5}$$

converges spherically uniformly on compact subsets of \mathbb{C} , where $g(\zeta)$ is a nonconstant meromorphic function on \mathbb{C} . From (5), we have that:

$$g_n^{(i)}(\zeta) = \rho_n^i f^{(i)}(z_n + \rho_n \zeta) \to g^{(i)}(\zeta)$$
(6)

converges uniformly on compact subsets of $\mathbb C$ disjoint from the poles of g.

Suppose that $g(\zeta_0) = 0$. Hurwitz's theorem implies that there exist ζ_n , $\zeta_n \to \zeta_0$ such that $f(z_n + \rho_n \zeta_n) = 0$. Since $\rho_n \to 0$, $z_n + \rho_n \zeta_n \in \Delta$ for sufficiently large *n*. Then by the assumptions given, we have $\max_{0 \le i \le k-1} |f^{(i)}(z_n + \rho_n \zeta_n)| \le M$. This and (6) imply that $g^{(i)}(\zeta_0) = 0$ for $0 \le i \le k-1$. Hence all zeros of *g*, if any, have multiplicity at least *k*. Moreover, $g^{(k)} \ne 0$.

Let $E = \{a_1, a_2, ..., a_{k+4}\}$, where $a_1, a_2, ..., a_{k+4}$ are distinct points in $\widehat{\mathbb{C}}$. Now suppose that $g(\zeta_0) = a_i$. By (5) and Hurwitz's theorem, there exists a sequence of points ζ_n , $\zeta_n \to \zeta_0$ such that $f(z_n + \rho_n \zeta_n) = a_i$. Obviously, $z_n + \rho_n \zeta_n \in \Delta$ for sufficiently large n, and then $z_n + \rho_n \zeta_n \in \Delta \cap f^{-1}(E)$. By the assumptions given, there exists a constant K > 0 such that for sufficiently large n:

$$\frac{1}{\varphi(|z_n+\rho_n\zeta_n|)^k}\frac{|f^{(k)}(z_n+\rho_n\zeta_n)|}{1+|f(z_n+\rho_n\zeta_n)|^{k+1}}\leqslant K.$$

It follows that:

$$\frac{|g_n^{(k)}(\zeta_n)|}{1+|g_n(\zeta_n)|^{k+1}} = \rho_n^k \frac{|f^{(k)}(z_n + \rho_n \zeta_n)|}{1+|f(z_n + \rho_n \zeta_n)|^{k+1}} \leqslant \left(\rho_n \varphi \left(|z_n + \rho_n \zeta_n|\right)\right)^k K.$$
⁽⁷⁾

Noting that φ is smoothly increasing and $\varphi(|z_n|)\rho_n \to 0$, we have $\varphi(|z_n + \rho_n \zeta_n|)/\varphi(|z_n|) \to 1$, so that:

$$\rho_n \varphi \big(|z_n + \rho_n \zeta_n| \big) = \varphi \big(|z_n| \big) \rho_n \frac{\varphi(|z_n + \rho_n \zeta_n|)}{\varphi(|z_n|)} \to 0 \quad \text{as } n \to \infty.$$

This, together with (7), leads to:

$$\frac{|g^{(k)}(\zeta_0)|}{1+|g(\zeta_0)|^{k+1}}=0.$$

Since $g^{(k)} \neq 0$, we conclude that ζ_0 is either the multiple pole of $g(\zeta)$ or the zero $g^{(k)}(\zeta)$. We thus have proved that if $g(\zeta_0) = a_i$, then ζ_0 is either a multiple pole of $g(\zeta)$ (for $a_i = \infty$) or a zero of $g^{(k)}(\zeta)$ (for $a_i \in \mathbb{C}$). This implies:

$$\sum_{i=1}^{k+4} \bar{N}\left(r, \frac{1}{g-a_i}\right) \leqslant \bar{N}_{(2}(r, g) + \bar{N}\left(r, \frac{1}{g^{(k)}}\right).$$

By Lemma 1, we have:

$$\begin{aligned} (k+2)T(r,g) &\leqslant \sum_{i=1}^{k+4} \bar{N}\left(r,\frac{1}{g-a_i}\right) + S(r,g) \leqslant \bar{N}_{(2}(r,g) + \bar{N}\left(r,\frac{1}{g^{(k)}}\right) + S(r,g) \\ &\leqslant \frac{1}{2}N(r,g) + T\left(r,g^{(k)}\right) + S(r,g) \leqslant \left(k + \frac{3}{2}\right)T(r,g) + S(r,g), \end{aligned}$$

that is, $\frac{1}{2}T(r, g) \leq S(r, g)$, which is a contradiction. Theorem 2 is thus proved. \Box

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