## Complex analysis

# Two results on $\varphi$-normal functions 

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## Deux résultats sur les fonctions $\varphi$-normales

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## A R T I C L E I N F O

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#### Abstract

In this paper, we obtain two results on $\varphi$-normal functions, which extend some related results due to Lappan, and Aulaskari-Rättyä. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

\section*{RÉS U M É}

Dans cette note, nous obtenons deux résultats sur les fonctions $\varphi$-normales, qui étendent des résultats connexes dus à Lappan et Aulaskari-Rättyä. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.


## 1. Introduction

Let $\Delta=\{z:|z|<1\}$ be the unit disc in the complex plane $\mathbb{C}$, and let $\mathcal{M}(\Delta)$ denote the set of all meromorphic functions in $\Delta$. A function $f \in \mathcal{M}(\Delta)$ is called a normal function, in the sense of Lehto and Virtanen [6], if

$$
\sup _{z \in \Delta}\left(1-|z|^{2}\right) f^{\#}(z)<\infty
$$

where $f^{\#}(z)=\left|f^{\prime}(z)\right| /\left(1+|f(z)|^{2}\right)$ is the spherical derivative of $f$.
An increasing function $\varphi:[0,1) \rightarrow(0, \infty)$ is called smoothly increasing if

$$
\varphi(r)(1-r) \rightarrow \infty, \quad \text { as } r \rightarrow 1^{-}
$$

and

$$
R_{a}(z)=\frac{\varphi(|a|+z / \varphi(|a|))}{\varphi(|a|)} \rightarrow 1, \quad \text { as }|a| \rightarrow 1^{-}
$$

uniformly on compact subsets of $\mathbb{C}$. For a given such $\varphi$, we call a function $f \in \mathcal{M}(\Delta)$ is $\varphi$-normal (see [1,2]) if

$$
\sup _{z \in \Delta} \frac{f^{\#}(z)}{\varphi(|z|)}<\infty
$$

Let $\mathcal{N}^{\varphi}$ denote the class of all $\varphi$-normal functions, and let $\mathcal{N}$ denote the class of all normal functions. Clearly, $\mathcal{N} \subset \mathcal{N}^{\varphi}$.

[^0]For a positive integer $k$, the expression $\left|f^{(k)}(z)\right| /\left(1+|f(z)|^{k+1}\right)$ can be viewed as an extension of the spherical derivative of $f$, which is introduced by Lappan [5]. In [5], Lappan also proved

Theorem A. Let $f \in \mathcal{M}(\Delta)$. If $f \in \mathcal{N}$, then for each positive integer $k$,

$$
\sup _{z \in \Delta}\left(1-|z|^{2}\right)^{k} \frac{\left|f^{(k)}(z)\right|}{1+|f(z)|^{k+1}}<\infty .
$$

The well-known Lappan five-point theorem [4] says that if $\sup \left\{\left(1-|z|^{2}\right) f^{\#}(z): z \in \Delta \cap f^{-1}(E)\right\}$ is bounded for some five-point $E$ subset of the extended plane $\widehat{\mathbb{C}}=\mathbb{C} \cup\{\infty\}$, then $f \in \mathcal{N}$. Recently, R. Aulaskari and J. Rättyä [2] got a version of Lappan five-point theorem for $\varphi$-normal functions, as follows.

Theorem B. Let $\varphi:[0,1) \rightarrow(0, \infty)$ be smoothly increasing, $k$ be a positive integer, and let $f \in \mathcal{M}(\Delta)$. If there exists a set $E$ of five distinct points in $\hat{\mathbb{C}}$ such that:

$$
\sup _{z \in \Delta \cap f^{-1}(E)} \frac{f^{\#}(z)}{\varphi(|z|)}<\infty
$$

then $f \in \mathcal{N}^{\varphi}$.
In this paper, we prove the following results.
Theorem 1. Let $\varphi:[0,1) \rightarrow(0, \infty)$ be smoothly increasing and $f \in \mathcal{M}(\Delta)$. If $f \in \mathcal{N}^{\varphi}$, then for each positive integer $k$,

$$
\sup _{z \in \Delta} \frac{1}{\varphi(|z|)^{k}} \frac{\left|f^{(k)}(z)\right|}{1+|f(z)|^{k+1}}<\infty
$$

Theorem 2. Let $\varphi:[0,1) \rightarrow(0, \infty)$ be smoothly increasing, $k$ be a positive integer, and let $f \in \mathcal{M}(\Delta)$, and suppose that there exists $M>0$ such that $\max _{0 \leqslant i \leqslant k-1}\left|f^{(i)}(z)\right| \leqslant M$ whenever $f(z)=0$ and $z \in \Delta$. If there exists a set $E$ of $k+4$ distinct points in $\hat{\mathbb{C}}$ such that:

$$
\sup _{z \in \Delta \cap f^{-1}(E)} \frac{1}{\varphi(|z|)^{k}} \frac{\left|f^{(k)}(z)\right|}{1+|f(z)|^{k+1}}<\infty
$$

then $f \in \mathcal{N}^{\varphi}$.
Remark. Clearly, Theorem 1 extends Theorem A, and our method to prove Theorem 1 is different from that in [5]. The condition " $\max _{0 \leqslant i \leqslant k-1}\left|f^{(i)}(z)\right| \leqslant M$ whenever $f(z)=0$ " in Theorem 2 holds naturally for $k=1$. So Theorem 2 is an extension of Lappan five-point theorem and Theorem B.

## 2. Lemmas

Let $f$ be a nonconstant meromorphic function in $\mathbb{C}$. We shall use the following standard notations of value distribution theory (see [3,8]):

$$
T(r, f), m(r, f), N(r, f), \bar{N}(r, f), \ldots
$$

We denote by $S(r, f)$ any function satisfying $S(r, f)=o\{T(r, f)\}, r \rightarrow \infty$, possibly outside a set with finite measure. We use $\bar{N}_{(2}(r, f)$ to denote the Nevanlinna counting function of the poles of $f$ with multiplicity $\geqslant 2$.

Lemma 1. (See [3,8].) Let $f$ be a nonconstant meromorphic function in $\mathbb{C}$, and let $a_{1}, a_{2}, \ldots, a_{q}(q \geqslant 3) \in \mathbb{C} \cup\{\infty\}$ be distinct complex numbers, and $k \in \mathbb{N}$. Then
(1) $(q-2) T(r, f) \leqslant \sum_{i=1}^{q} \bar{N}\left(r, \frac{1}{f-a_{i}}\right)+S(r, f)$.
(2) $T\left(r, f^{(k)}\right) \leqslant(k+1) T(r, f)+S(r, f)$.

The next lemma reveals a close relationship between $\varphi$-normal functions and normal families, which is a direct consequence of Marty's theorem; it can be founded in [1,2].

Lemma 2. Let $\varphi:[0,1) \rightarrow(0, \infty)$ be smoothly increasing, and let $f \in \mathcal{M}(\Delta)$. Then $f \in \mathcal{N}^{\varphi}$ if and only if the family $\{f(a+$ $z / \varphi(|a|)): a \in \Delta\}$ is a normal in $\Delta$.

The following is a version of Lohwater-Pommerenke theorem [7] for $\mathcal{N}^{\varphi}$ (see [1,2]).
Lemma 3. Let $\varphi:[0,1) \rightarrow(0, \infty)$ be smoothly increasing, and $f \in \mathcal{M}(\Delta)$. If $f \notin \mathcal{N}^{\varphi}$, then there exist a sequence of points $z_{n} \in D$, two sequences of positive numbers $\rho_{n}, \sigma_{n}$ with $\sigma_{n} \rightarrow 0$, and a constant $c>0$ satisfying $\varphi\left(\left|z_{n}\right|\right) \rho_{n} \leqslant c \sigma_{n}$ such that $f\left(z_{n}+\rho_{n} \zeta\right)$ spherically and uniformly converges to a nonconstant meromorphic function on each compact subset of $\mathbb{C}$.

## 3. Proof of theorems

Proof of Theorem 1. Theorem 1 is true for $k=1$ by the definition of the $\varphi$-normal function. Suppose that Theorem 1 is not true for $k \geqslant 2$, then there exists a sequence $\left\{z_{n}\right\} \subset \Delta$ such that:

$$
\begin{equation*}
\frac{1}{\varphi\left(\left|z_{n}\right|\right)^{k}} \frac{\left|f^{(k)}\left(z_{n}\right)\right|}{1+\left|f\left(z_{n}\right)\right|^{k+1}} \rightarrow \infty, \quad n \rightarrow \infty \tag{1}
\end{equation*}
$$

Set the family:

$$
\mathcal{G}=\left\{g_{n}(z)=f\left(z_{n}+z / \varphi\left(\left|z_{n}\right|\right)\right)\right\}
$$

By Lemma $2, \mathcal{G}$ is a normal family in $\Delta$. Then, for each sequence $\left\{g_{n}\right\} \in \mathcal{G}$, there exists a subsequence of $\left\{g_{n}\right\}$ (without loss of generality, we still denote by $\left\{g_{n}\right\}$ for convenience) such that $g_{n}(z) \rightarrow g(z)$ converges spherically locally uniformly in $\Delta$, where $g(z)$ is a meromorphic function (possibly infinity identically).

We distinguish two cases.
Case 1. $g(z) \equiv \infty$. Then $1 / g_{n} \rightarrow 0$ in $\Delta$, and thus $\left(1 / g_{n}\right)^{(i)} \rightarrow 0$ for positive integer $i$. In particular, $g_{n}^{\prime} / g_{n}^{2}=-\left(1 / g_{n}\right)^{\prime} \rightarrow 0$. On the other hand, an elementary calculation yields:

$$
\frac{g_{n}^{(k)}}{g_{n}^{k+1}}=-\frac{1}{g_{n}^{k-1}}\left(\frac{1}{g_{n}}\right)^{(k)}+P\left(\frac{g_{n}^{\prime}}{g_{n}^{2}}, \frac{g_{n}^{\prime \prime}}{g_{n}^{3}}, \ldots, \frac{g_{n}^{(k-1)}}{g_{n}^{k}}\right)
$$

where $P\left(w_{1}, w_{2}, \ldots, w_{k-1}\right)$ is a polynomial in $w_{1}, w_{2}, \ldots, w_{k-1}$ with integer coefficients. By induction, we have $\frac{g_{n}^{(k)}(z)}{g_{n}^{k+1}(z)} \rightarrow 0$ in $\Delta$. It follows that:

$$
\begin{equation*}
\frac{\left|g_{n}^{(k)}(z)\right|}{1+\left|g_{n}(z)\right|^{k+1}} \leqslant\left|\frac{g_{n}^{(k)}(z)}{g_{n}^{k+1}(z)}\right| \rightarrow 0 \tag{2}
\end{equation*}
$$

in $\Delta$. Note that:

$$
\begin{equation*}
\frac{\left|g_{n}^{(k)}(0)\right|}{1+\left|g_{n}(0)\right|^{k+1}}=\frac{1}{\varphi\left(\left|z_{n}\right|\right)^{k}} \frac{\left|f^{(k)}\left(z_{n}\right)\right|}{1+\left|f\left(z_{n}\right)\right|^{k+1}} \tag{3}
\end{equation*}
$$

Then, from (2) and (3), we have:

$$
\frac{1}{\varphi\left(\left|z_{n}\right|\right)^{k}} \frac{\left|f^{(k)}\left(z_{n}\right)\right|}{1+\left|f\left(z_{n}\right)\right|^{k+1}} \rightarrow 0, \quad \text { as } n \rightarrow \infty
$$

which contradicts (1).
Case 2. $g(z) \not \equiv \infty$. Now we divide into subcases.
Case 2.1. $g(0) \neq \infty$.
Then there exists $0<\delta<1$ such that $g(z)$ is holomorphic in $\Delta_{\delta}=\{z:|z|<\delta\}$, and hence $g_{n}(z)$-for $n$ sufficiently large-are holomorphic. Since $g_{n}(z) \rightarrow g(z)$, we get:

$$
\frac{\left|g_{n}^{(k)}(z)\right|}{1+\left|g_{n}(z)\right|^{k+1}} \rightarrow \frac{\left|g^{(k)}(z)\right|}{1+|g(z)|^{k+1}}, \quad z \in \Delta_{\delta}
$$

Letting $M_{1}=\left|g^{(k)}(0)\right| /\left(1+|g(0)|^{k+1}\right)$, then for sufficiently large $n$ :

$$
\frac{\left|g_{n}^{(k)}(0)\right|}{1+\left|g_{n}(0)\right|^{k+1}} \leqslant M_{1}+1
$$

This and (3) give:

$$
\frac{1}{\varphi\left(\left|z_{n}\right|\right)^{k}} \frac{\left|f^{(k)}\left(z_{n}\right)\right|}{1+\left|f\left(z_{n}\right)\right|^{k+1}} \leqslant M_{1}+1
$$

which is a contradiction.
Case 2.2. $g(0)=\infty$.
Then we can find $0<\delta<1$ such that $g(z)$ is holomorphic and $|g(z)| \geqslant 2$ in $\Delta_{\delta}^{\prime}=\{z: 0<|z|<\delta\}$, and thus $g_{n}(z)$ is holomorphic and $\left|g_{n}(z)\right| \geqslant 2$ in $\Delta_{\delta}^{\prime}$ for sufficiently large $n$. So we know that:

$$
\frac{g^{(k)}(z)}{1+g^{k+1}(z)} \quad \text { and } \quad \frac{g_{n}^{(k)}(z)}{1+g_{n}^{k+1}(z)} \quad \text { (for sufficiently large } n \text { ) }
$$

are holomorphic in $\Delta_{\delta}$, and

$$
\begin{equation*}
\frac{g_{n}^{(k)}(z)}{1+g_{n}^{k+1}(z)} \rightarrow \frac{g^{(k)}(z)}{1+g^{k+1}(z)}, \quad z \in \Delta_{\delta}^{\prime} \tag{4}
\end{equation*}
$$

The maximum modulus principle implies that (4) still holds in $\Delta_{\delta}$. Letting $M_{2}=\left|g^{(k)}(0)\right| /\left|1+|g(0)|^{k+1}\right|$, for sufficiently large $n$, we have:

$$
\frac{\left|g_{n}^{(k)}(0)\right|}{1+\left|g_{n}(0)\right|^{k+1}} \leqslant\left|\frac{g_{n}^{(k)}(0)}{1+g_{n}(0)^{k+1}}\right| \leqslant M_{2}+1
$$

As in Case 2.1, we arrive at a contradiction. This completes the proof of Theorem 1.
Proof of Theorem 2. Suppose that $f \notin \mathcal{N}^{\varphi}$. Then, by Lemma 3, there exist points $z_{n} \in \Delta$ tending to the boundary, positive numbers $\rho_{n}$ with $\varphi\left(\left|z_{n}\right|\right) \rho_{n} \rightarrow 0$ such that:

$$
\begin{equation*}
g_{n}(\zeta)=f\left(z_{n}+\rho_{n} \zeta\right) \rightarrow g(\zeta) \tag{5}
\end{equation*}
$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\zeta)$ is a nonconstant meromorphic function on $\mathbb{C}$. From (5), we have that:

$$
\begin{equation*}
g_{n}^{(i)}(\zeta)=\rho_{n}^{i} f^{(i)}\left(z_{n}+\rho_{n} \zeta\right) \rightarrow g^{(i)}(\zeta) \tag{6}
\end{equation*}
$$

converges uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of $g$.
Suppose that $g\left(\zeta_{0}\right)=0$. Hurwitz's theorem implies that there exist $\zeta_{n}, \zeta_{n} \rightarrow \zeta_{0}$ such that $f\left(z_{n}+\rho_{n} \zeta_{n}\right)=0$. Since $\rho_{n} \rightarrow 0$, $z_{n}+\rho_{n} \zeta_{n} \in \Delta$ for sufficiently large $n$. Then by the assumptions given, we have $\max _{0 \leqslant i \leqslant k-1}\left|f^{(i)}\left(z_{n}+\rho_{n} \zeta_{n}\right)\right| \leqslant M$. This and (6) imply that $g^{(i)}\left(\zeta_{0}\right)=0$ for $0 \leqslant i \leqslant k-1$. Hence all zeros of $g$, if any, have multiplicity at least $k$. Moreover, $g^{(k)} \neq 0$.

Let $E=\left\{a_{1}, a_{2}, \ldots, a_{k+4}\right\}$, where $a_{1}, a_{2}, \ldots, a_{k+4}$ are distinct points in $\widehat{\mathbb{C}}$. Now suppose that $g\left(\zeta_{0}\right)=a_{i}$. By (5) and Hurwitz's theorem, there exists a sequence of points $\zeta_{n}, \zeta_{n} \rightarrow \zeta_{0}$ such that $f\left(z_{n}+\rho_{n} \zeta_{n}\right)=a_{i}$. Obviously, $z_{n}+\rho_{n} \zeta_{n} \in \Delta$ for sufficiently large $n$, and then $z_{n}+\rho_{n} \zeta_{n} \in \Delta \cap f^{-1}(E)$. By the assumptions given, there exists a constant $K>0$ such that for sufficiently large $n$ :

$$
\frac{1}{\varphi\left(\left|z_{n}+\rho_{n} \zeta_{n}\right|\right)^{k}} \frac{\left|f^{(k)}\left(z_{n}+\rho_{n} \zeta_{n}\right)\right|}{1+\left|f\left(z_{n}+\rho_{n} \zeta_{n}\right)\right|^{k+1}} \leqslant K
$$

It follows that:

$$
\begin{equation*}
\frac{\left|g_{n}^{(k)}\left(\zeta_{n}\right)\right|}{1+\left|g_{n}\left(\zeta_{n}\right)\right|^{k+1}}=\rho_{n}^{k} \frac{\left|f^{(k)}\left(z_{n}+\rho_{n} \zeta_{n}\right)\right|}{1+\left|f\left(z_{n}+\rho_{n} \zeta_{n}\right)\right|^{k+1}} \leqslant\left(\rho_{n} \varphi\left(\left|z_{n}+\rho_{n} \zeta_{n}\right|\right)\right)^{k} K \tag{7}
\end{equation*}
$$

Noting that $\varphi$ is smoothly increasing and $\varphi\left(\left|z_{n}\right|\right) \rho_{n} \rightarrow 0$, we have $\varphi\left(\left|z_{n}+\rho_{n} \zeta_{n}\right|\right) / \varphi\left(\left|z_{n}\right|\right) \rightarrow 1$, so that:

$$
\rho_{n} \varphi\left(\left|z_{n}+\rho_{n} \zeta_{n}\right|\right)=\varphi\left(\left|z_{n}\right|\right) \rho_{n} \frac{\varphi\left(\left|z_{n}+\rho_{n} \zeta_{n}\right|\right)}{\varphi\left(\left|z_{n}\right|\right)} \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

This, together with (7), leads to:

$$
\frac{\left|g^{(k)}\left(\zeta_{0}\right)\right|}{1+\left|g\left(\zeta_{0}\right)\right|^{k+1}}=0
$$

Since $g^{(k)} \not \equiv 0$, we conclude that $\zeta_{0}$ is either the multiple pole of $g(\zeta)$ or the zero $g^{(k)}(\zeta)$. We thus have proved that if $g\left(\zeta_{0}\right)=a_{i}$, then $\zeta_{0}$ is either a multiple pole of $g(\zeta)$ (for $a_{i}=\infty$ ) or a zero of $g^{(k)}(\zeta)$ (for $a_{i} \in \mathbb{C}$ ). This implies:

$$
\sum_{i=1}^{k+4} \bar{N}\left(r, \frac{1}{g-a_{i}}\right) \leqslant \bar{N}_{(2}(r, g)+\bar{N}\left(r, \frac{1}{g^{(k)}}\right)
$$

By Lemma 1, we have:

$$
\begin{aligned}
(k+2) T(r, g) & \leqslant \sum_{i=1}^{k+4} \bar{N}\left(r, \frac{1}{g-a_{i}}\right)+S(r, g) \leqslant \bar{N}_{(2}(r, g)+\bar{N}\left(r, \frac{1}{g^{(k)}}\right)+S(r, g) \\
& \leqslant \frac{1}{2} N(r, g)+T\left(r, g^{(k)}\right)+S(r, g) \leqslant\left(k+\frac{3}{2}\right) T(r, g)+S(r, g)
\end{aligned}
$$

that is, $\frac{1}{2} T(r, g) \leqslant S(r, g)$, which is a contradiction. Theorem 2 is thus proved.

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