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Algebra

Relations of Keune symbols

Relations entre symboles de Keune

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ABSTRACT

In this paper, we extend the symbols that Keune defined for commutative rings in 1981 to the non-commutative case and discuss their relations. These relations are similar to those of the commutative case.

RÉSUMÉ

Dans cet article, nous étendons la définition des symboles introduits par Keune en 1981 pour les anneaux commutatifs au cas des anneaux non commutatifs, et nous étudions leurs relations. Elles sont similaires à celles du cas commutatif.

1. Introduction

Let R be a ring with identity. We knew that, for a ring with stable range one R, there is an exact sequence:

$$1 \to K_2(R) \to U_R \to G \to 1,$$

where U_R is a group defined by generators and relations (see [3]), its generators with the form $\langle a,b,c,d\rangle$ $(1+ab+ad+cd+abcd \in U(R))$, and G is a subgroup of unit group of R. If R is commutative ring, and $a,b,c \in R$ with $a+c-abc=1$, Keune defined:

$$\langle a,b,c\rangle = x_{ij}(-a)x_{ji}(b)x_{ij}(-c)x_{ji}(1-ab)x_{ij}\bigl(-(1-bc)\bigr)w_{ij}(1)$$

in [1] and gave the presentation of $K_2(R)$ for commutative rings with stable range one based on such kind of symbols in [2]. In this paper, we give the definition of $\langle a,b,c\rangle$ for a general ring with identity along the direction of Keune, and some relations of this symbols are discussed. These relations are similar to those of [1].

2. Some relations in the group $K(R)$

In the following, R is always a ring with identity 1, $U(R)$ its unit group, φ means the homomorphism $St_n(R) \to E_n(R), x_{ij}(r) \mapsto e_{ij}(r)$ for $n \geqslant 3$. If i,j are different positive integrals, for $a,b,u \in R$ with $u, \beta = 1-ba \in U(R)$, definitions of $w_{ij}(u)$, $h_{ij}(u)$, $H_{ij}(a,b)$, a Steinberg symbol $\{u,v\}_1 = h_{1j}(u)h_{1j}(v)h_{1j}(vu)^{-1}$, and a Dennis–Stein symbol $\langle a,b\rangle_1 = H_{1j}(a,b)h_{1j}^{-1}(\beta)$ are as usual (see [3]).

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By a simple computation, we have: if $u = a + c - abc \in U(R)$, then $v = a + c - cba \in U(R)$.

Definition 2.1. Let $a, b, c \in R$, put $u = a + c - abc$, $v = a + c - cba$. If $u \in U(R)$, define:

$$W_{ij}(a,b,c) = x_{ij}(-a)x_{ji}(b)x_{ij}(-c)x_{ji}\big(u^{-1}(1-ab)\big)x_{ij}\big(-v(1-bc)\big),$$
$$\langle a,b,c\rangle_i = W_{ij}(a,b,c)w_{ij}(v).$$

$\langle a,b,c\rangle_i$ is called a *Keune symbol*. Let $\mathcal{W}_n(R)$ be the subgroup of $St_n(R)$ generated by all elements $W_{ij}(a,b,c)$, and let $K(R)$ be the subgroup of $St_n(R)$ generated by all Keune symbols $\langle a,b,c\rangle_1$.

The following lemma is frequently used, but its proof is similar to that of Lemma 9.2 of [4].

Lemma 2.2. *Conjugation by an element of $\mathcal{W}_n(R)$ carries each generator $x_{ij}(r)$ of $St_n(R)$ into some generator $x_{ij}(s)$ of $St_n(R)$.*

Note that $\langle a,b,c\rangle_i$ is independent of the choice of j by Lemma 2.2. It is easy to see $w_{ij}(u), H_{ij}(a,b) \in \mathcal{W}_n(R)$.

Properties of $W_n(R)$ are described in the following proposition. Its proof uses Lemma 2.2 and is similar to the proof of Theorem 1.5 of [1], though (iii) is much more complicated.

Proposition 2.3. *The followings hold in $\mathcal{W}_n(R)$.*

(i) $W_{ij}^{-1}(a,b,c) = W_{ij}(-c,-b,-a)$.
(ii) $W_{ij}(a,b,c) = W_{ji}(-(1-bc)u^{-1},-a,-b)$, $W_{ij}(a,b,c) = W_{ji}(-b,-c,-u^{-1}(1-ab))$.
(iii) $W_{ij}^{-1}(a,bc,d)W_{jk}^{-1}(b,ca,e)W_{ki}^{-1}(c,ab,f) = w_{jk}(v)$, *where* $f = s^{-1}r^{-1}(1-(db+re)c)$, $r = a+d(1-bca)$, $s = b+e(1-cab)$, $v = b + (1-bca)e$.
(iv) $W_{ji}(a,b,(1-ac)d)W_{kj}(av^{-1},vc,d(1-ba)v^{-1}) = w_{jk}(1)W_{ki}(a,b+c-bac,d)$, *where* $v = a + (1-ac)d(1-ba) \in U(R)$.

In order to discuss relations for Keune's symbols, we introduce first the following lemma.

Lemma 2.4. *Let $a,b,d \in R$ with $u = a + d - abd \in U(R)$, $v = a + d - dba$. Then*:

$$\langle a,b,d\rangle_k = W_{ji}(a,b,d)w_{ki}(-u)w_{kj}(-1)w_{ki}(v) = W_{ji}(-a,-b,-d)w_{ki}(-u)w_{kj}(1)w_{ki}(v)$$
$$= W_{ji}(-a,-b,-d)w_{ki}(u)w_{kj}(-1)w_{ki}(-v).$$

Proof. By Definition 2.1, we have $W_{kj}(av^{-1},0,d(1-ba)v^{-1}) = w_{kj}(-1)$. Let $c = 0$ in Proposition 2.3(iv), $W_{ki}(a,b,d) = W_{ji}(a,b,d)w_{ki}(-u)w_{kj}(-1)$. So $\langle a,b,d\rangle_k = W_{ji}(a,b,d)w_{ki}(-u)w_{kj}(-1)w_{ki}(v)$.

If $l \neq k,i,j$, by Lemma 2.2, we have $\langle a,b,d\rangle_k = h_{jl}(-1)\langle a,b,d\rangle_k h_{jl}(-1)^{-1} = W_{ji}(-a,-b,-d)w_{ki}(-u)w_{kj}(1)w_{ki}(v)$. Similarly, we can also get $\langle a,b,d\rangle_k = W_{ji}(-a,-b,-d)w_{ki}(u)w_{kj}(-1)w_{ki}(-v)$. □

In the following, $\langle ua,bu^{-1},uc\rangle_1$ is denoted by ${}^{u}\langle a,b,c\rangle_1$.

Now we begin to discuss the relations of Keune symbols; these relations are similar to those of [1].

Theorem 2.5. *Let R be a ring. Then the following equations hold in $K(R)$*:

(i) *for $h \in St_n(R)$ with $\varphi(h) = diag(\alpha_1,\dots,\alpha_n)$, $h\langle a,b,c\rangle_1 h^{-1} = {}^{\alpha_1}\langle a,b,c\rangle_1$.*
(ii) $\langle a,b,c\rangle_1 = \langle c,b,a\rangle_1^{-1}$.
(iii) $\langle a,b,c\rangle_1 = \langle ub,cu^{-1},(1-ab)\rangle_1$.
(iv) $\langle a,b+c-bac,d\rangle_1 = \langle a,b,(1-ac)d\rangle_1 \cdot \langle a,c,d(1-ba)\rangle_1$.
(v) $\langle a,b,c\rangle_1 = \langle a\gamma,\gamma^{-1}b,c\gamma\rangle_1$ *for $\gamma \in U(R)$.*
(vi) $x\langle a,b,c\rangle_1 x^{-1} = {}^{\pi}\langle a,b,c\rangle_1$ *for all $x \in K(R)$, where $\pi = \varphi(x)$.*

Proof. (i), (v) and (vi) From Lemma 2.2, $h\langle a,b,c\rangle_1 h^{-1} = hW_{1j}(a,b,c)w_{1j}(v)h^{-1} = \langle \alpha_1 a\alpha_j^{-1}, \alpha_j b\alpha_1^{-1}, \alpha_1 c\alpha_j^{-1}\rangle_1$. By this formula, (vi) is obvious.

If $l \neq 1, j$, we have $\langle a,b,c\rangle_1 = h_{lj}(\gamma)\langle a,b,c\rangle_1 h_{lj}^{-1}(\gamma)$. $h_{lj}(\gamma)$ take the place of h, we have $\langle a,b,c\rangle_1 = \langle a\gamma,\gamma^{-1}b,c\gamma\rangle_1$. So (v) has been proved. From (v), we get (i).

(ii) Put $u = a + c - abc \in U(R)$, $v = a + c - cba$. Noting that $w_{1i}(v) = w_{1i}^{-1}(-v)$ and $w_{1i}(u)^{-1} = w_{i1}(u^{-1})$, by Lemma 2.4 and Lemma 2.2, we have:

$$\langle a,b,c\rangle_1^{-1} = w_{1i}(-v)w_{1j}(1)W_{ji}(-c,-b,-a) = W_{ji}(-c,-b,-a)w_{1j}\big(vu^{-1}\big)w_{1i}(v)w_{1j}(-1)$$
$$= \langle c,b,a\rangle_1 w_{1i}(u)w_{1j}(1)w_{1i}(-v)w_{1j}\big(vu^{-1}\big)w_{1i}(v)w_{1j}(-1) = \langle c,b,a\rangle_1.$$

(iii) If $l \neq 1$, and $u = a + c - abc \in U(R)$, then by Lemma 2.2, Lemma 2.4, Proposition 2.3(ii), we have:

$$\begin{aligned}\langle a,b,c\rangle_1 &= h_{il}(u)\langle a,b,c\rangle_1 h_{il}(u)^{-1} = h_{il}(u)W_{ji}(a,b,c)w_{1i}(-u)w_{1j}(-1)w_{1i}(v)h_{il}(u)^{-1}\\ &= h_{il}(u)W_{ij}\big(-b,-c,u^{-1}(1-ab)\big)w_{1i}(-u)w_{1j}(-1)w_{1i}(v)h_{il}(u)^{-1}\\ &= W_{ij}\big(-ub,-cu^{-1},-(1-ab)\big)w_{1i}(-1)w_{1j}(-1)w_{1i}\big(vu^{-1}\big)\\ &= \big\langle ub,cu^{-1},(1-ab)\big\rangle_1 w_{1j}(1)w_{1i}(1)w_{1j}\big(-uv^{-1}\big)w_{1i}(-1)w_{1j}(-1)w_{1i}\big(vu^{-1}\big)\\ &= \big\langle ub,cu^{-1},(1-ab)\big\rangle_1.\end{aligned}$$

(iv) By Definition 2.1, Proposition 2.3(iv), Lemma 2.4, we have:

$$\begin{aligned}\langle a,b+c-bac,d\rangle_1 &= W_{1i}(a,b+c-bac,d)w_{1i}(w)\\ &= W_{ji}\big(a,b,(1-ac)d\big)w_{1i}(-u)W_{1j}\big(av^{-1},vc,d(1-ba)v^{-1}\big)w_{1i}(w)\\ &= \big\langle a,b,(1-ac)d\big\rangle_1 w_{1i}(-v)w_{1j}(1)W_{1j}\big(av^{-1},vc,d(1-ba)v^{-1}\big)w_{1i}(w)\\ &= \big\langle a,b,(1-ac)d\big\rangle_1 W_{ji}\big(-a,-c,-d(1-ba)\big)w_{1i}(-v)w_{1j}(1)w_{1i}(w)\\ &= \big\langle a,b,(1-ac)d\big\rangle_1\cdot\big\langle a,c,d(1-ba)\big\rangle_1. \qquad \square\end{aligned}$$

Proposition 2.6. (iv) *is equivalent to the following under* (ii), (iii), (v), (vi) *in Theorem* 2.5,

(vii) *Let* $a,b,c,d,e \in R$, *put* $\begin{cases} u = a+(1-abc)d,\\ r = a+d(1-bca),\end{cases}$ $\begin{cases} v = b+(1-bca)e,\\ s = b+e(1-cab).\end{cases}$ *If* $u,v \in U(R)$, *and* $f = s^{-1}r^{-1}(1-(db+re)c)$, *then* $\langle a,bc,d\rangle_1 \cdot {}^{r}\langle b,ca,e\rangle_1 = \langle ab,c,dv+ae\rangle_1$, *or* $\langle a,bc,d\rangle_1\cdot{}^{r}\langle b,ca,e\rangle_1\cdot{}^{rs}\langle c,ab,f\rangle_1 = 1$.

Proof. (iv) ⇒ (vii) It is obvious that $\langle a,bc,d\rangle\cdot{}^{r}\langle b,ca,e\rangle_1 = \langle ab,c,dv+ae\rangle_1$ is equivalent to $\langle b,ca,e\rangle_1\cdot{}^{u^{-1}}\langle a,bc,d\rangle_1 = {}^{u^{-1}}\langle ab,c,dv+ae\rangle_1$ from (i). By a simple computation, we have $u^{-1}(1-abc) = (1-bca)r^{-1}$, $1-(1-bca)ev^{-1} = bv^{-1}$, $1-r^{-1}d(1-bca) = r^{-1}a$, $r^{-1}d+ev^{-1}-r^{-1}d(1-bca)ev^{-1} = r^{-1}(dv+ae)v^{-1}$, $u^{-1}(1-abc)+vc-u^{-1}(1-abc)(dv+ae)c = u^{-1}$ and $1-u^{-1}(1-abc)(dv+ae)v^{-1} = u^{-1}abv^{-1}$. By (ii), we have:

$$\begin{aligned}\langle b,ca,e\rangle_1\cdot{}^{u^{-1}}\langle a,bc,d\rangle_1 &= \big\langle vca,ev^{-1},1-bca\big\rangle_1\cdot\big\langle bc,d,u^{-1}(1-abc)\big\rangle_1\\ &= \big\langle vca,ev^{-1},1-bca\big\rangle_1\cdot\big\langle bcr,r^{-1}d,1-bca\big\rangle_1\\ &= \big(\big\langle 1-bca,r^{-1}d,bv^{-1}\cdot vcr\big\rangle_1\cdot\big\langle 1-bca,ev^{-1},vcr\cdot r^{-1}a\big\rangle_1\big)^{-1}\\ &= \big(\big\langle 1-bca,r^{-1}d+ev^{-1}-r^{-1}d(1-bca)ev^{-1},vcr\big\rangle_1\big)^{-1}\\ &= \big(\big\langle 1-bca,r^{-1}(dv+ae)v^{-1},vcr\big\rangle_1\big)^{-1} = \big(\big\langle (1-bca)r^{-1},(dv+ae)v^{-1},vc\big\rangle_1\big)^{-1}\\ &= \big(\big\langle u^{-1}(dv+ae)v^{-1},vcu,u^{-1}abv^{-1}\big\rangle_1\big)^{-1} = \big(\big\langle u^{-1}(dv+ae),cu,u^{-1}ab\big\rangle_1\big)^{-1}\\ &= {}^{u^{-1}}\langle ab,c,dv+ae\rangle_1.\end{aligned}$$

(vii) ⇒ (iv) Let $p = a+(1-ab)(1-ac)d$, $q = a+(1-ac)d(1-ba)$, $w = a+d(1-ba)(1-ca)$. Then by (iii) we have:

$$\begin{aligned}\big\langle a,b,(1-ac)d\big\rangle_1\cdot\big\langle a,c,d(1-ba)\big\rangle_1 &= \big\langle pb,(1-ac)dp^{-1},1-ab\big\rangle_1\cdot\big\langle qc,d(1-ba)q^{-1},1-ac\big\rangle_1\\ &= \big(\big\langle 1-ac,d(1-ba)q^{-1},qc\big\rangle_1\cdot\big\langle 1-ab,(1-ac)dp^{-1},pb\big\rangle_1\big)^{-1}\\ &= \big(\big\langle 1-ab,(1-ac)dp^{-1},pb\big\rangle_1\cdot{}^{pq^{-1}}\big\langle 1-ac,dp^{-1}(1-ab),qc\big\rangle_1\big)^{-1}\\ &= \big(\big\langle (1-ab)(1-ac),dp^{-1},pb+(1-ab)qc\big\rangle_1\big)^{-1}\\ &= \big(\big\langle dp^{-1},p(b+c-bac),1-(1-ab)(1-ac)dp^{-1}\big\rangle_1\big)^{-1}\\ &= \big(\langle d,b+c-bac,a\rangle_1\big)^{-1} = \langle a,b+c-bac,d\rangle_1. \qquad \square\end{aligned}$$

Let $S(R)$ be the subgroup of $St_n(R)$ generated by all Steinberg symbols $\{\alpha,\beta\}_1$ of R, and $D(R)$ be the subgroup generated by all Dennis–Stein symbols $\langle a,b\rangle_1$ of R.

The following proposition shows that Steinberg symbols and Dennis–Stein symbols are special Keune symbols.

Proposition 2.7. *Let R be a ring.*

(i) *If $\alpha, \beta \in U(R)$, then $\{\alpha, \beta\}_1 = \langle \alpha - \alpha^2, \alpha^{-1}, \alpha + \beta - 1\rangle_1$.*
(ii) *If $a, b \in R$ with $1 - ab \in U(R)$, $\beta = 1 - ba$, then $\langle a, b\rangle_1 = \langle a - 1, -1, b - 1\rangle_1$.*

Proof. (i) By definition of $\{\alpha, \beta\}_1$, we have:

$$\begin{aligned}\{\alpha, \beta\}_1 &= w_{1j}(\alpha)w_{1j}(-1)w_{1j}(\beta)w_{1j}(-\beta\alpha)\\ &= x_{j1}\big(\alpha^{-2}\big)x_{1j}\big(-\alpha^2\big)w_{1j}(\alpha)x_{1j}(-1)w_{1j}(\beta)w_{1j}(-\beta\alpha)\\ &= x_{1j}\big(-\alpha^2\big)w_{1j}(\alpha)x_{1j}(-1)w_{1j}(\beta)w_{1j}(-\beta\alpha)x_{j1}\big(\alpha^{-1}\beta\alpha^{-1}\beta^{-1}\big)\\ &= x_{1j}\big(-\alpha^2\big)w_{1j}(\alpha)x_{1j}(-1)w_{1j}(\beta)x_{1j}\big(-\beta^2\big)w_{1j}(-\beta\alpha)\\ &= x_{1j}\big(-\alpha^2+\alpha\big)x_{j1}\big(-\alpha^{-1}\big)x_{1j}(\alpha+\beta-1)x_{j1}\big(-\beta^{-1}\big)x_{1j}\big(-\beta^2+\beta\big)w_{1j}(-\beta\alpha)\\ &= \big\langle\alpha^2-\alpha, -\alpha^{-1}, 1-\alpha-\beta\big\rangle_1 = \big\langle\alpha-\alpha^2, \alpha^{-1}, \alpha+\beta-1\big\rangle_1.\end{aligned}$$

(ii) By definition of $\langle a, b\rangle_i$, we have:

$$\begin{aligned}\langle a, b\rangle_1 &= x_{1j}(-a)x_{j1}(b)x_{1j}\big(\alpha^{-1}a\big)x_{j1}(-\beta b)w_{1j}(1)w_{1j}(-\beta)\\ &= x_{1j}(-a)w_{1j}(1)x_{1j}(-b)x_{j1}\big(-\alpha^{-1}a\big)x_{1j}(\beta b)w_{1j}(-\beta)\\ &= x_{1j}\big(-(a-1)\big)w_{j1}(-1)x_{1j}\big(-(b-1)\big)x_{j1}\big(-\alpha^{-1}a\big)x_{1j}(\beta b)w_{1j}(-\beta)\\ &= \langle a-1, -1, b-1\rangle_1.\end{aligned}$$

So $S(R) \subseteq D(R) \subseteq K(R)$. □

Remark 2.8. In [1], an example is given of a ring R for which $S(R) \subsetneq D(R)$ and also one for which $D(R) \subsetneq K(R)$.

If $a, b, c \in R$, and $u = a + c - abc \in U(R)$, $v = a + c - cba$. Put:

$$H_{ij}(a, b, c) = W_{ij}(a, b, c)w_{ij}(1).$$

Let $\mathcal{H}_n(R)$ be the subgroup of $St_n(R)$ generated by all $H_{ij}(a, b, c)$ where $a, b, c \in R$ with $a + c - abc \in U(R)$. If $a, b, u \in R$ with $1 - ab, u \in U(R)$, we have $h_{ij}(u) = H_{ij}(1 - u, 1)$, $H_{ij}(a, b) = H_{ij}(a + 1, 1, b + 1) \in \mathcal{H}_n(R)$.

Proposition 2.9. *Let R be a ring. Then any element $h \in \mathcal{H}_n(R)$ is uniquely written as $h = xh_{12}(u_2)\cdots h_{1n}(u_n)$ where $x \in K(R)$, $u_2, u_3, \ldots, u_n \in U(R)$.*

Proof. At first, the Steinberg symbol $\{\alpha, \beta\}_1$ and the Dennis–Stein symbol $\langle a, b\rangle_1$ belong to $K(R)$. Secondly, by Definition 2.1, we have $H_{1j}(a, b, c) = \langle a, b, c\rangle_1 h_{j1}(v^{-1})$, and $H_{j1}(a, b, c) = \langle -b, -c, -u^{-1}(1 - ab)\rangle_1 h_{1j}(-u^{-1})$. Thirdly, if $a, b, c \in R$ with $u = a + c - abc \in U(R)$, $v = a + c - cba$. Then, by Lemma 2.4 and a simple computation, we have $H_{ij}(a, b, c) = \langle a, b, c\rangle_1 h_{1j}(-v)h_{i1}(-u)h_{ji}(-1)$.

This shows that $\mathcal{H}_n(R)$ is generated by all elements $\langle a, b, c\rangle_1$ and $h_{ij}(u)$, where $u \in U(R)$. From Section 9 of [4], we have $h_{ij}(u) = h_{1j}(u)h_{i1}(u)$, $h_{i1}(u) = h_{1i}^{-1}(u) = \{u, u\}_1 h_{1i}(u^{-1})$, $h_{1i}(u)h_{1j}(v) = \{u, v\}_1 h_{1j}(v)h_{1i}(u)$, and $h_{1i}(u)h_{1i}(v) = \{u, v\}_1 h_{1i}(vu)$, so by Theorem 2.5(i), we get the result. □

The proof of the following proposition is similar to the proof of Theorem 4.2 in [1].

Proposition 2.10. *Let R be a ring with stable range one. Then the structure of $St_n(R)$ is $St_n(R) = LUL\mathcal{H}_n(R)$, where L is the subgroup generated by $x_{ij}(r)$ with $1 \leqslant j < i \leqslant n$, $r \in R$, U is the subgroup generated by $x_{ij}(r)$ with $1 \leqslant i < j \leqslant n$, $r \in R$.*

Corollary 2.11. *Let R be a ring with stable range one. Then there is a short exact sequence*:

$$1 \to K_2(R) \to K(R) \to W \to 1,$$

where $K(R)$ is the group generated by all elements $\langle a, b, c\rangle_1 = W_{1j}(a, b, c)w_{1j}(v)$ and W is a subgroup of unit group of R generated by all elements $(a + c - abc)(a + c - cba)^{-1}$.

3. General Keune symbols

Let R be a ring. Put K_R to be the group with the following presentation:

Generators: $\langle a,b,c\rangle$, where $a,b,c\in I$ with $a+c-abc\in U(R)$;

Relations: $(K1)$ $\langle a,b,c\rangle^{-1}=\langle c,b,a\rangle$,

$(K2)$ $\langle a,b,d\rangle=\langle ub,du^{-1},(1-ab)\rangle$,

$(K3)$ $\langle a,b+c-bac,d\rangle=\langle a,b,(1-ac)d\rangle\cdot\langle a,c,d(1-ba)\rangle$,

$(K4)$ $\langle a,b,c\rangle=\langle a\gamma^{-1},\gamma b,c\gamma^{-1}\rangle$ for $\gamma\in U(R)$,

$(K5)$ $\langle A,B,C\rangle\langle a,b,c\rangle\langle A,B,C\rangle^{-1}={}^{x}\langle a,b,c\rangle$,

where $x=(A+C-ABC)(A+C-CBA)^{-1}$ and ${}^{x}\langle a,b,c\rangle=\langle xa,bx^{-1},xc\rangle$. $\langle a,b,c\rangle$ is called a general Keune symbol.

The groups D_R and S_R are defined by generators and relations (see [3]): generators of D_R are of the form $\langle a,b\rangle$ with $1-ab\in U(R)$, relations of D_R are:

$(D1)$ $\langle a,b\rangle^{-1}=\langle b,a\rangle$,
$(D2)$ $\langle b+c-bac,a\rangle={}^{1-ba}\langle c,a\rangle\cdot\langle b,a\rangle$,
$(D3)$ $\langle a,bc\rangle\langle b,ca\rangle\langle c,ab\rangle=1$,
$(D4)$ $\langle a,b\rangle\langle c,d\rangle\langle a,b\rangle^{-1}={}^{\pi}\langle c,d\rangle$, where $\pi=(1-ab)(1-ba)^{-1}$;

generators of S_R are of the form $\{u,v\}$ with $u,v\in U(R)$, relations of S_R are:

$(S1)$ $\{u,1-u\}=1$,
$(S2)$ $\{u,-u\}=1$,
$(S3)$ $\{u,vw\}=\{u,v\}\cdot{}^{v}\{u,w\}$,
$(S4)$ $\{uv,w\}={}^{u}\{v,w\}\{u,w\}$,

where ${}^{u}a=uau^{-1}$, ${}^{u}\{v,w\}=\{{}^{u}v,{}^{u}w\}$, ${}^{u}\langle a,b\rangle=\langle{}^{u}a,{}^{u}b\rangle$.

We call $\langle a,b\rangle$ a general Dennis–Stein symbol and $\{u,v\}$ a general Steinberg symbol.

A simple computation shows that $\langle a,b,c\rangle=1$ if one of a,b,c is zero.

In the following, we discuss the relation between general Dennis–Stein symbols, general Steinberg symbols and general Keune symbols.

Proposition 3.1. *The maps*:

$$D_R\to K_R,\quad \langle a,b\rangle\mapsto\langle a-1,-1,b-1\rangle\quad\textit{and}\quad S_R\to K_R,\quad \{u,v\}\mapsto\langle u-u^2,u^{-1},u+v-1\rangle$$

are group homomorphisms.

Proof. We do not distinguish between $\langle a,b\rangle$ and its image, and between $\{u,v\}$ and its image. So put $\langle a,b\rangle=\langle a-1,-1,b-1\rangle$ and $\{u,v\}=\langle u-u^2,u^{-1},u+v-1\rangle$.

$(D1)$ $\langle a,b\rangle^{-1}=\langle a-1,-1,b-1\rangle^{-1}=\langle b-1,-1,a-1\rangle=\langle b,a\rangle$ by $(K1)$.

$(D3)$ Let $u=1-cab$, $v=1-bca$, $w=1-abc$. Then $a=1-(1-a)u^{-1}u$, $uv^{-1}b=1-u(1-b)u^{-1}$. By $(D1)$, $(K2)$, $(K3)$, $(K5)$, we have:

$$\begin{aligned}\big(\langle a,bc\rangle\langle b,ca\rangle\big)^{-1}&=\langle ca,b\rangle\langle bc,a\rangle=\langle ca-1,-1,b-1\rangle\langle bc-1,-1,a-1\rangle\\&=\langle u,(1-b)u^{-1},ca\rangle\langle v,(1-a)v^{-1},bc\rangle=\langle u,(1-a)u^{-1},uv^{-1}bc\rangle\langle u,(1-b)u^{-1},ca\rangle\\&=\langle u,(1-ab)u^{-1},c\rangle=\langle c-1,-1,ab-1\rangle=\langle c,ab\rangle.\end{aligned}$$

$(D4)$ Let $\pi=(1-ab)(1-ba)^{-1}$. Then by $(K4)$, $(K5)$ we get:

$$\begin{aligned}\langle a,b\rangle\langle c,d\rangle\langle a,b\rangle^{-1}&=\langle a-1,-1,b-1\rangle\langle c-1,-1,d-1\rangle\langle a-1,-1,b-1\rangle^{-1}\\&=\langle\pi(c-1),-\pi^{-1},\pi(d-1)\rangle=\langle\pi c\pi^{-1}-1,-1,\pi d\pi^{-1}-1\rangle={}^{\pi}\langle c,d\rangle.\end{aligned}$$

For the sake of proving $(D2)$, we first prove the following lemma.

Lemma 3.2. *Let $a,b,u,v\in R$ with $\alpha=1-ab$, $u,v\in U(R)$. Then*

(1) $\langle a,b\rangle=\langle a,b,(1-a)\beta^{-1}\rangle$.

(2) $\{u,v\} = \langle (1-u)v^{-1}, v\rangle = \langle v^{-1}, v(1-u)\rangle$.
(3) $\{u,v\}\{v,u\} = 1$.

Proof. (1) Let $\alpha = 1-ab$, $\beta = 1-ba$, $\pi = (1-ba)(1-ab)^{-1}$. Then $\alpha a = a\beta$ and $\beta b = b\alpha$. From $(K2)$, $(K4)$, $(K5)$, we have:

$$\begin{aligned}\langle a,b\rangle &= \langle a-1,-1,b-1\rangle = \langle \alpha\beta^{-1}(1-b)\alpha^{-1}, a\beta\alpha^{-1}, \alpha b\alpha^{-1}\rangle = \langle a,b\rangle\langle 1-b, a\beta^{-1}, \beta b\rangle\langle a,b\rangle^{-1}\\ &= \langle a,b\rangle\langle \alpha a\beta^{-1}, \beta b\alpha^{-1}, (1-a)\beta^{-1}\rangle\langle a,b\rangle^{-1} = \langle a,b\rangle\langle a,b,(1-a)\beta^{-1}\rangle\langle a,b\rangle^{-1}\\ &= \langle a,b,(1-a)\beta^{-1}\rangle\langle {}^{\pi}a, {}^{\pi}b\rangle\langle a,b\rangle^{-1} = \langle a,b,(1-a)\beta^{-1}\rangle.\end{aligned}$$

(2) From $(K1)$, $(K2)$ and (1), we get:

$$\{u,v\}^{-1} = \langle u+v-1, u^{-1}, u-u^2\rangle = \langle v, (1-u)v^{-1}, (1-v)u^{-1}\rangle = \langle v, (1-u)v^{-1}\rangle = \langle (1-u)v^{-1}, v\rangle^{-1}.$$

So we have $\{u,v\} = \langle (1-u)v^{-1}, v\rangle$. Similarly, $\{u,v\} = \langle v^{-1}, v(1-u)\rangle$.

(3) Note that $1 - v^{-1}(u+v-1) = v^{-1}(1-u)$, $1-(u+v-1)u^{-1} = (1-v)u^{-1}$, $v - v^2 = (1-v)u^{-1}uv$, and $u-u^2 = uvv^{-1}(1-u)$, by $(K3)$, $(K2)$, we have:

$$\begin{aligned}\big(\{u,v\}\{v,u\}\big)^{-1} &= \langle u+v-1, v^{-1}, v-v^2\rangle\langle u+v-1, u^{-1}, u-u^2\rangle\\ &= \langle u+v-1, v^{-1}+u^{-1}-v^{-1}(u+v-1)u^{-1}, uv\rangle\\ &= \langle u+v-1, v^{-1}u^{-1}, uv\rangle = \langle 1, 1-(u+v-1)v^{-1}u^{-1}, 0\rangle = 1.\end{aligned}$$

Now we prove $(D2)$. Let $\gamma = 1-ab$, $\delta = 1-ba$, $\eta = 1-ac$, $\theta = 1-ca$, $\alpha = 1-a(b+c-bac) = \gamma\eta$, $\beta = 1-(b+c-bac)a = \delta\theta$. Then by Lemma 3.2, $(K3)$, $(K5)$ we have:

$$\begin{aligned}\langle b+c-bac, a\rangle^{-1} &= \langle a, b+c-bac, \big((1-a)\beta^{-1}\big)\rangle = \langle a,b,(1-ac)(1-a)\beta^{-1}\rangle\langle a,c,(1-a)\beta^{-1}(1-ba)\rangle\\ &= \langle a,b,(1-ac)(1-a)\beta^{-1}\rangle\langle a,c,(1-a)\theta^{-1}\rangle = \langle a,b,(1-ac)(1-a)\beta^{-1}\rangle\langle a,c\rangle\\ &= \langle a,c\rangle\langle \theta\eta^{-1}a, b\eta\theta^{-1}, \theta\eta^{-1}(1-ac)(1-a)\theta^{-1}\delta^{-1}\rangle = \langle a,c\rangle\langle \theta\eta^{-1}a, b\eta\theta^{-1}, \theta(1-a)\theta^{-1}\delta^{-1}\rangle\\ &= \langle a,c\rangle\langle \theta\eta^{-1}a, b\eta\theta^{-1}\rangle = \langle a\theta\eta^{-1}, \eta\theta^{-1}b\rangle\langle a,c\rangle = \big(\langle c,a\rangle\langle \eta\theta^{-1}b, a\theta\eta^{-1}\rangle\big)^{-1}.\end{aligned}$$

So we have $\langle b+c-bac, a\rangle = \langle c,a\rangle\langle \eta\theta^{-1}b, a\theta\eta^{-1}\rangle$. By $(D3)$, we have $\langle \eta\theta^{-1}b, a\theta\eta^{-1}\rangle = \langle \eta\theta^{-1}ba, \theta\eta^{-1}\rangle\langle b,a\rangle = \{\eta\theta^{-1}, \delta\}\langle b,a\rangle$, and

$$\begin{aligned}\langle c,a\rangle &= {}^{\delta^{-1}}\langle \delta c\delta^{-1}, \delta a\delta^{-1}\rangle = \langle \delta c\delta^{-1}, \delta a\delta^{-1}\rangle\{\delta\eta\theta^{-1}\delta^{-1}, \delta^{-1}\}\\ &= \langle \delta c\delta^{-1}, \delta a\delta^{-1}\rangle\cdot {}^{\delta^{-1}}\{\eta\theta^{-1}, \delta^{-1}\} = \langle \delta c\delta^{-1}, \delta a\delta^{-1}\rangle\{\delta, \eta\theta^{-1}\}.\end{aligned}$$

Now we get $\langle b+c-bac, a\rangle = \langle {}^{1-ba}c, {}^{1-ba}a\rangle\cdot\langle b,a\rangle$.

The homomorphism $S_R \longrightarrow D_R$ is treated in [3] and $S_R \longrightarrow K_R$ is the composition of $S_R \longrightarrow D_R$ and $D_R \longrightarrow K_R$. □

Remark 3.3. Recently, the authors have proved that for rings R with stable range one, the group $K(R)$ is isomorphic to K_R.

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References

[1] F. Keune, Another presentation for the K_2 of a local domain, J. Pure Appl. Algebra 22 (1981) 131–141.
[2] F. Keune, The K_2 of a 1-Fold Stable Ring, Lect. Notes Math., vol. 1046, Springer-Verlag, Berlin, 1984, pp. 193–228.
[3] M. Kolster, General symbols and presentations of elementary linear groups, J. Reine Angew. Math. 353 (1984) 132–164.
[4] J. Milnor, Introduction to Algebraic K-Theory, Ann. Math. Stud., vol. 72, Princeton University Press, Princeton, 1971.