Partial differential equations/Differential geometry

# Maximum principles and isoperimetric inequalities for some Monge-Ampère-type problems 

# Principes du maximum et inégalités isopérimétriques pour certains problèmes du type Monge-Ampère 

Cristian Enache

Research group of the project PN-II-ID-PCE-2012-4-0021, "Simion Stoilow" Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

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#### Abstract

In this note we derive a maximum principle for an appropriate functional combination of $u(\mathbf{x})$ and $|\nabla u|^{2}$, where $u(\mathbf{x})$ is a strictly convex classical solution to a general class of Monge-Ampère equations. This maximum principle is then employed to establish some isoperimetric inequalities of interest in the theory of surfaces of constant Gauss curvature in $\mathbb{R}^{N+1}$. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É Dans cette note, nous obtenons un principe du maximum pour une combinaison fonctionnelle appropriée de $u(\mathbf{x})$ et $|\nabla u|^{2}$, où $u(\mathbf{x})$ est une solution classique strictement convexe à une classe générale d'équations du type Monge-Ampère. Ce principe du maximum est ensuite utilisé pour établir certaines inégalités isopérimétriques d'intérêt dans la théorie de surfaces de courbure de Gauss constante dans $\mathbb{R}^{N+1}$.


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## 1. Introduction and main results

Let $\Omega \subset \mathbb{R}^{N}, N \geqslant 2$, be a bounded strictly convex $C^{2}$ domain. This note deals with the following general class of MongeAmpère equations:

$$
\begin{equation*}
\operatorname{det}\left(D^{2} u\right)=f(u) g\left(|\nabla u|^{2}\right) \quad \text { in } \Omega \tag{1.1}
\end{equation*}
$$

where $f$ and $g$ are some real positive $C^{1}$ functions, with $f^{\prime} \geqslant 0$. We also assume that Eq. (1.1) is uniformly elliptic, i.e. we impose throughout the strong ellipticity condition $A^{*}:=\left(S_{N}^{i j}\right)>0$ in $\Omega$, where $S_{N}^{i j}:=\partial\left(\operatorname{det}\left(D^{2} u\right)\right) / \partial u_{i j}$, which means that a solution to Monge-Ampère equation (1.1) is assumed throughout to be strictly convex in $\Omega$. Under these assumptions, Hopf's first maximum principle [5] implies that a classical solution $u(\mathbf{x})$ to Eq. (1.1) assumes its minimum value on $\partial \Omega$.

Let us now consider the following auxiliary function, which is a kind $P$-function in the sense of L.E. Payne (see the book of R. Sperb [15] or the paper of G.A. Philippin and S. Vernier-Piro [12]):

[^0]\[

$$
\begin{equation*}
P(\mathbf{x}, \alpha):=\int_{0}^{|\nabla u|^{2}} \rho(y) \mathrm{d} y-2 \alpha \int_{0}^{u} f(s)^{\frac{1}{N}} \mathrm{~d} s, \quad \text { with } \alpha \in[0,1] \tag{1.2}
\end{equation*}
$$

\]

where $u(\mathbf{x})$ is a classical solution of Eq. (1.1) and:

$$
\begin{equation*}
\rho(y):=\frac{1}{g(y)}\left(\frac{N}{2} y^{-\frac{N}{2}} \int_{0}^{y} \frac{1}{g(s)} s^{\frac{N}{2}-1} \mathrm{~d} s\right)^{\frac{1}{N}-1} . \tag{1.3}
\end{equation*}
$$

The main result of this note states:
Theorem 1.1. Assume that $u(\mathbf{x}) \in C^{3}(\Omega) \cap C^{2}(\bar{\Omega})$ is a strictly convex solution of Eq. (1.1) and $P(\mathbf{x}, \alpha)$ is the auxiliary function defined in (1.2). Then:
i) $P(\mathbf{x}, 0)$ takes its maximum value on $\partial \Omega$;
ii) if $g^{\prime} \geqslant 0$ and $\alpha \in(0,1], P(\mathbf{x}, a)$ takes its maximum value on $\partial \Omega$.

We note that particular cases of Theorem 1.1 have been considered and investigated in some previous works, namely in X.-N. Ma [8] (the case $N=2, f \equiv g \equiv 1$; see, also, C. Enache [3] for a complementary result), G.A. Philippin and A. Safoui [11] (the case $f \equiv g \equiv 1$ ), C. Enache [4] (the case $N=2$ ), respectively. Furthermore, G.A. Philippin and A. Safoui [10] (see, also, C. Enache [2] or L. Barbu and C. Enache [1]) have also investigated the general class of Monge-Ampère equations (1.1) and derived a similar maximum principle for a different auxiliary function, namely for:

$$
\begin{equation*}
\Phi(\mathbf{x}):=\int_{0}^{|\nabla u|^{2}} g^{-\frac{1}{N}}(s) \mathrm{d} s-2 \int_{0}^{u} f^{\frac{1}{N}}(s) \mathrm{d} s \tag{1.4}
\end{equation*}
$$

under the following additional assumption on the data $f$ and $g$ :

$$
\begin{equation*}
g^{-\frac{1}{N}} \frac{f^{\prime}}{f}+2 f \frac{1}{N} \frac{g^{\prime}}{g} \geqslant 0 \tag{1.5}
\end{equation*}
$$

More precisely, they proved that $\Phi(\mathbf{x})$ takes its maximum value on $\partial \Omega$. However, as we will notice later (see Remark 1 in Section 2), the maximum principle stated in Theorem 1.1 for $P(\mathbf{x}, 1)$ is the best possible when $f \equiv$ const. This means that $P(\mathbf{x}, 1)$ satisfies a maximum principle and that there exists a domain of optimality on which $P$ is identically constant. Thus, when Eq. (1.1) is subject to a Dirichlet boundary condition and $f \equiv$ const., our maximum principle may be employed to derive isoperimetric inequalities. Conversely, $\Phi(\mathbf{x})$ cannot be identically constant for $f \not \equiv$ const. or $g \not \equiv$ const., so that the maximum principle derived by G.A. Philippin and A. Safoui [10] is not the best possible in such a case.

As applications of Theorem 1.1, we are going to establish some isoperimetric inequalities of interest in the theory of surfaces of constant Gauss curvature. To this end, we will investigate the particular case $f \equiv k_{0}=$ const. $>0, g(s)=$ $(1+s)^{(N+2) / 2}$ in (1.1), when $k_{0}$ represents the Gauss curvature of the hypersurface $x_{N+1}=u\left(x_{1}, \ldots, x_{N}\right)$ in the Euclidean space $\mathbb{R}^{N+1}$. More precisely, let us consider the following problem:

$$
\left\{\begin{array}{l}
\operatorname{det}\left(D^{2} u\right)=k_{0}\left(1+|\nabla u|^{2}\right)^{\frac{N+2}{2}}>0 \text { in } \Omega  \tag{1.6}\\
u=0 \text { on } \partial \Omega
\end{array}\right.
$$

Moreover, assume that the (Gauss) curvature $G(s)$ of $\partial \Omega$ satisfies the following existence criterion (see N.M. Ivochkina [7]):

$$
\begin{equation*}
G \geqslant k_{0}^{\frac{N-1}{N}} \quad \text { on } \partial \Omega \tag{1.7}
\end{equation*}
$$

Our second result states:
Theorem 1.2. Let $u \in C^{3}(\Omega) \cap C^{2}(\bar{\Omega})$ be the strictly convex solution of problem (1.6) and denote

$$
\begin{align*}
& u_{\min }:=\min _{\bar{\Omega}} u(\mathbf{x}), \quad S:=\left\{(\mathbf{x}, u(\mathbf{x})): \mathbf{x}=\left(x_{1}, \ldots, x_{N}\right) \in \Omega\right\}, \\
& \left.A:=\int_{\Omega} \sqrt{1+|\nabla u|^{2}} \mathrm{~d} x \quad \text { (the area of the surface } S\right), \\
& V:=-\int_{\Omega} u \mathrm{~d} x \quad \text { (the volume between } \Omega \text { and the surface } S \text { ). } \tag{1.8}
\end{align*}
$$

We then have the following estimates:

$$
\begin{equation*}
u_{\min } \geqslant \frac{1}{k_{0}^{\frac{1}{N}}}\left(\frac{\sqrt{G_{\min }^{\frac{2}{N-1}}-k_{0}^{\frac{2}{N}}}}{G_{\min }^{\frac{1}{N-1}}}-1\right) \tag{1.9}
\end{equation*}
$$

respectively

$$
\begin{equation*}
(N+1) k_{0}^{\frac{1}{N}} V \leqslant A \tag{1.10}
\end{equation*}
$$

The equality sign holds in the above inequalities if and only if $\Omega$ is a ball and $S$ is an $N$-dimensional hemisphere.
We note that isoperimetric estimate (1.9) is sharper than an isoperimetric estimate obtained by H. Rosenberg in [13], which states the following: a compact graph, with positive constant Gauss curvature $k_{0}$ in $R^{N+1}$ and boundary situated in an $N$-dimensional hyperplane, can reach at most a height $1 / k_{0}^{\frac{1}{N}}$, this height being attained only by the $N$-dimensional hemisphere of radius $1 / k_{0}^{\frac{1}{N}}$. We also note that isoperimetric inequality (1.10) states, in other words, the following: if $\Omega$ is a bounded strictly convex $C^{2}$ domain in $R^{N}(N \geqslant 2)$ satisfying (1.7), on which by (1.6) it is possible to define a nonparametric surface $S$ of prescribed constant Gauss curvature $k_{0}>0$ and of prescribed area $A$, then the volume $V$ bounded by $S$ and $\Omega$ is greatest when $\Omega$ is a ball and $S$ is an $N$-dimensional hemisphere. Finally, we mention that similar isoperimetric estimates for surfaces of constant mean curvature have been derived in L.E. Payne and G.A. Philippin in [9].

Theorem 1.1 and Theorem 1.2 will be proved successively in Section 2 and Section 3, respectively.
For convenience, we note that throughout the paper the comma is used to indicate differentiation, the summation from 1 to $N$ is understood on repeated indices and, when appropriate, we use the following notations:

$$
\begin{align*}
& A:=D^{2} u=\left(u_{i j}\right), \quad\left(u^{i j}\right):=A^{-1}, \quad R:=\left(u_{i} u_{j}\right), \\
& u_{n}:=\frac{\partial u}{\partial n} \quad(\text { the outward normal derivative of } u(\mathbf{x}) \text { on } \partial \Omega), \\
& u_{\min }:=\min _{\bar{\Omega}} u(\mathbf{x}), \quad q_{\max }:=\max _{\partial \Omega}|\nabla u(\mathbf{x})|, \\
& \left.G_{\min }:=\min _{\partial \Omega} G(s) \quad \text { (the minimum value of the (Gauss) curvature } G(s) \text { on } \partial \Omega\right) . \tag{1.11}
\end{align*}
$$

Also, when the arguments of $f(s), g(y), \rho(y)$ will be omitted in a formula, this means that these functions are evaluated at $s=u$ and $y=|\nabla u|^{2}$, respectively.

## 2. The proof of Theorem 1.1

### 2.1. The proof of Theorem 1.1.i)

Differentiating successively (1.2), we obtain:

$$
\begin{align*}
& P_{k}=2 \rho u_{i k} u_{i}-2 \alpha f^{\frac{1}{N}} u_{k}  \tag{2.1}\\
& u^{k l} P_{k l}=4 \rho^{\prime} u^{k l} u_{i k} u_{i} u_{j l} u_{j}+2 \rho u^{k l} u_{i k l} u_{i}+2 \rho u^{k l} u_{i k} u_{i l}-2 \alpha \frac{1}{N} f^{\frac{1}{N}-1} f^{\prime} u^{k l} u_{k} u_{l}-2 \alpha f^{\frac{1}{N}} u^{k l} u_{k l} \tag{2.2}
\end{align*}
$$

We compute separately each term of (2.2). First, we note that:

$$
\begin{equation*}
u^{k l} u_{i k} u_{i} u_{j l} u_{j}=\operatorname{tr}\left(A^{-1} A R A\right)=\operatorname{tr}(R A)=u_{i j} u_{i} u_{j} \tag{2.3}
\end{equation*}
$$

so that, using Eq. (2.1) to replace the term $u_{i j} u_{i} u_{j}$ in (2.3), we get:

$$
\begin{equation*}
u^{k l} u_{i k} u_{i} u_{j l} u_{j}=\alpha f^{\frac{1}{N}} \frac{1}{\rho}|\nabla u|^{2}+\text { terms containing } P_{k} . \tag{2.4}
\end{equation*}
$$

To compute $u^{k l} u_{i k l} u_{i}$, we differentiate Eq. (1.1) and obtain:

$$
\begin{align*}
u^{k l} u_{k l i} u_{i} & =\frac{1}{\operatorname{det}\left(D^{2} u\right)} \frac{\partial\left(\operatorname{det}\left(D^{2} u\right)\right)}{\partial u_{k l}} \frac{\partial u_{k l}}{\partial x_{i}} u_{i} \\
& =\frac{1}{\operatorname{det}\left(D^{2} u\right)} \frac{\partial\left(\operatorname{det}\left(D^{2} u\right)\right)}{\partial x_{i}} u_{i} \\
& =\frac{1}{f g}\left(f^{\prime} g|\nabla u|^{2}+2 f g^{\prime} u_{i k} u_{i} u_{k}\right) . \tag{2.5}
\end{align*}
$$

Therefore, using Eq. (2.1) to replace the term $u_{i k} u_{i} u_{k}$ in (2.5), we obtain:

$$
\begin{equation*}
u^{k l} u_{k l i} u_{i}=\frac{f^{\prime}}{f}|\nabla u|^{2}+2 \alpha f^{\frac{1}{N}} \frac{g^{\prime}}{g} \frac{1}{\rho}|\nabla u|^{2}+\text { terms containing } P_{k} . \tag{2.6}
\end{equation*}
$$

Next, we note that:

$$
\begin{equation*}
u^{k l} u_{i k} u_{i l}=\operatorname{tr}\left(A^{-1} A^{2}\right)=\operatorname{tr}(A)=\Delta u \tag{2.7}
\end{equation*}
$$

so that, making use of the following inequality:

$$
\begin{equation*}
\frac{\Delta u}{N} \geqslant \operatorname{det}\left(D^{2} u\right)^{\frac{1}{N}} \tag{2.8}
\end{equation*}
$$

we have:

$$
\begin{equation*}
u^{k l} u_{i k} u_{i l} \geqslant N f^{\frac{1}{N}} g^{\frac{1}{N}} \tag{2.9}
\end{equation*}
$$

Inserting now (2.4), (2.6) and (2.9) into (2.2), we have:

$$
\begin{align*}
L P(\mathbf{x} .0) & :=u^{k l} P_{k l}(\mathbf{x}, 0)+\text { terms containing } P_{k}(\mathbf{x}, 0) \\
& =2 \rho\left[\frac{f^{\prime}}{f}|\nabla u|^{2}+N f^{\frac{1}{N}} g^{\frac{1}{N}}\right]>0 \quad \text { in } \Omega \tag{2.10}
\end{align*}
$$

since $f^{\prime} \geqslant 0$ and $f^{\frac{1}{N}} g^{\frac{1}{N}}>0$. Therefore, Hopf's first maximum principle [5] implies that $P(\mathbf{x}, 0)$ attains its maximum value on the boundary $\partial \Omega$. The proof of Theorem 1.1.i) is thus achieved.

### 2.2. The proof of Theorem 1.1.ii)

First, we note that:

$$
\begin{equation*}
u^{k l} u_{k l}=\operatorname{tr}\left(A A^{-1}\right)=\operatorname{tr}\left(I_{N}\right)=N \tag{2.11}
\end{equation*}
$$

Then, making use of (2.1), we have:

$$
\begin{align*}
u^{k l} u_{k} u_{l} & =\frac{\rho}{\alpha} f^{-\frac{1}{N}} u^{k l} u_{i k} u_{i} u_{l}+\text { terms containing } P_{k} \\
& =\frac{\rho}{\alpha} f^{-\frac{1}{N}} \operatorname{tr}\left(A^{-1} A R\right)+\text { terms containing } P_{k} \\
& =\frac{\rho}{\alpha} f^{-\frac{1}{N}}|\nabla u|^{2}+\text { terms containing } P_{k} \tag{2.12}
\end{align*}
$$

Therefore, the insertion of (2.4), (2.6), (2.9), (2.11) and (2.12) into (2.2) leads to:

$$
\begin{align*}
L P(\mathbf{x}, \alpha): & =u^{k l} P_{k l}(\mathbf{x}, \alpha)+\text { terms containing } P_{k}(\mathbf{x}, \alpha) \\
\geqslant & 2 f^{\frac{1}{N}}\left\{N\left(\rho g^{\frac{1}{N}}-\alpha\right)+2 \alpha[\log (\rho g)]^{\prime}|\nabla u|^{2}\right\} \\
& +2\left(1-\frac{1}{N}\right) \frac{f^{\prime}}{f} \rho|\nabla u|^{2}+\text { terms containing } P_{k}(\mathbf{x}, \alpha) \text { in } \Omega . \tag{2.13}
\end{align*}
$$

Now, from the definition of $\rho$ and the fact that $g^{\prime} \geqslant 0$, we have:

$$
\begin{equation*}
\rho g^{\frac{1}{N}} \geqslant 1 \tag{2.14}
\end{equation*}
$$

respectively

$$
\begin{equation*}
2(\log (\rho g))^{\prime}|\nabla u|^{2} \geqslant 0 \tag{2.15}
\end{equation*}
$$

Making now use (2.14) and (2.15) in (2.13), we obtain:

$$
\begin{equation*}
L P \geqslant 0 \quad \text { in } \Omega . \tag{2.16}
\end{equation*}
$$

Consequently, Hopf's first maximum principle [5] implies that $P(\mathbf{x}, \alpha)$ attains its maximum value on the boundary $\partial \Omega$, unless it is constant on $\bar{\Omega}$. The proof of Theorem 1.1.ii) is thus achieved.

Remark 1. If $u=u(r)$ is a solution of the ordinary differential equation:

$$
\begin{equation*}
\left(\frac{N}{2} \int_{0}^{u^{\prime}(r)^{2}} \frac{1}{g(s)} s^{\frac{N}{2}-1} \mathrm{~d} s\right)^{\frac{1}{N}}=r \tag{2.17}
\end{equation*}
$$

then, differentiating (2.17), we obtain:

$$
\begin{equation*}
\frac{u^{\prime \prime}\left(u^{\prime}\right)^{N-1}}{g\left(u^{\prime}(r)^{2}\right)\left[\left(\frac{N}{2} \int_{0}^{u^{\prime}(r)^{2}} \frac{1}{g(s)} s^{\frac{N}{2}-1} \mathrm{~d} s\right)^{\frac{1}{N}}\right]^{N-1}}=1 \tag{2.18}
\end{equation*}
$$

Making use of (2.17) in (2.18), we conclude that $u$ is in fact a radial solution of Eq. (1.1), with $f \equiv 1$, i.e.

$$
\begin{equation*}
\operatorname{det}\left(D^{2} u\right)=g\left(u^{\prime}(r)^{2}\right), \quad \text { with } r:=|\mathbf{x}| \tag{2.19}
\end{equation*}
$$

(which is the case, for instance, when $\Omega$ is a ball and $u$ satisfies the Dirichlet boundary condition $u \equiv$ const. on $\partial \Omega$; see G.A. Philippin and A. Safoui [10] or C. Enache [2]). The corresponding $P$-function $P(r, 1)$ reads in this case:

$$
\begin{equation*}
P(r, 1):=\int_{0}^{u^{\prime}(r)^{2}} \frac{1}{g(y)}\left(\frac{N}{2} y^{-\frac{N}{2}} \int_{0}^{y} \frac{1}{g(s)} s^{\frac{N}{2}-1} \mathrm{~d} s\right)^{\frac{1}{N}-1} \mathrm{~d} y-2 u . \tag{2.20}
\end{equation*}
$$

Differentiating (2.20) and making use (2.18), we obtain:

$$
\begin{equation*}
\frac{\mathrm{d} P}{\mathrm{~d} r}=0 \tag{2.21}
\end{equation*}
$$

which means that Theorem 1.1 yields a best-possible maximum principle for $P(\mathbf{x}, 1)$.

## 3. The proof of Theorem 1.2

3.1. The proof of inequality (1.9)

With $f \equiv k_{0}$ and $g(s)=(1+s)^{\frac{N+2}{2}}$ in (1.1), Theorem 1.1.ii) implies that the auxiliary function $P(\mathbf{x}, 1)$, given in this case as:

$$
\begin{equation*}
P(\mathbf{x}, 1):=2\left(1-\frac{1}{\sqrt{1+|\nabla u|^{2}}}-k_{0}^{\frac{1}{N}} u\right) \tag{3.1}
\end{equation*}
$$

takes its maximum value at some point $\mathbf{Q} \in \partial \Omega$, unless it is constant on $\bar{\Omega}$. Therefore, we have:

$$
\begin{equation*}
-k_{0}^{\frac{1}{N}} u \leqslant \frac{1}{\sqrt{1+|\nabla u|^{2}}}-\frac{1}{\sqrt{1+q_{\max }^{2}}} \tag{3.2}
\end{equation*}
$$

We notice here that, by Theorem 1.1.i), $q_{\max }$ coincides with the maximum value of $|\nabla u|$ on $\bar{\Omega}$. Evaluating now (3.2) at the unique critical point of $u(\mathbf{x})$, we obtain:

$$
\begin{equation*}
-k_{0}^{\frac{1}{N}} u_{\min } \leqslant 1-\frac{1}{\sqrt{1+q_{\max }^{2}}} \tag{3.3}
\end{equation*}
$$

Next, we would like to construct a lower bound for $q_{\max }$ in terms of the (Gauss) curvature of $\partial \Omega$. To this end, we first note that, since $P(\mathbf{x}, 1)$ takes its maximum value at the point $\mathbf{Q} \in \partial \Omega$, we have $\partial P(\mathbf{Q}, 1) / \partial n \geqslant 0$, or:

$$
\begin{equation*}
\frac{2 u_{n} u_{n n}}{\left(1+u_{n}^{2}\right)^{3 / 2}}-2 k_{0}^{\frac{1}{N}} u_{n} \geqslant 0 \quad \text { at } \mathbf{Q} . \tag{3.4}
\end{equation*}
$$

On the other hand, since the boundary $\partial \Omega$ is smooth, Eq. (1.6) may be evaluated on $\partial \Omega$, in normal coordinates with respect to $\partial \Omega$, as it follows (see Lemma 8 in G.A. Philippin and A. Safoui [11]):

$$
\begin{equation*}
G u_{n n} u_{n}^{N-1}=k_{0}\left(1+u_{n}^{2}\right)^{\frac{N+2}{2}} \quad \text { on } \partial \Omega, \tag{3.5}
\end{equation*}
$$

where $G$ is the (Gauss) curvature of $\partial \Omega$. Inserting (3.5) in (3.4) and taking into account that $u_{n}>0$ on $\partial \Omega$ (due to Hopf's second maximum principle [6]), we get:

$$
\begin{equation*}
q_{\max }=|\nabla u|(\mathbf{O}) \leqslant \frac{k_{0}^{\frac{1}{N}}}{\sqrt{G^{\frac{2}{N-1}}-k_{0}^{\frac{2}{N}}}} \leqslant \frac{k_{0}^{\frac{1}{N}}}{\sqrt{G_{\min }^{\frac{2}{N-1}}-k_{0}^{\frac{2}{N}}}} \tag{3.6}
\end{equation*}
$$

Therefore, making use of (3.6) in (3.3), we obtain (1.9).

### 3.2. The proof of inequality (1.10)

From (3.2) we have:

$$
\begin{equation*}
-k_{0}^{\frac{1}{N}} u \leqslant \frac{1}{\sqrt{1+|\nabla u|^{2}}}=\sqrt{1+|\nabla u|^{2}}-\frac{|\nabla u|^{2}}{\sqrt{1+|\nabla u|^{2}}} \tag{3.7}
\end{equation*}
$$

With $h(s):=(1+s)^{-1 / 2}$ let us introduce the following matrix:

$$
\begin{equation*}
E:=\left(h I_{2}+2 h^{\prime} R\right) A \tag{3.8}
\end{equation*}
$$

Then, the eigenvalues of $E$ are exactly the curvatures $k_{1}, \ldots, k_{N}$ of the surface $x_{N+1}=u(\mathbf{x})$. Moreover, since $u(\mathbf{x})$ is strictly convex, $k_{i}>0$ for all $i=1, \ldots, N$. Therefore, the matrix $E$ is positive definite and we have:

$$
\begin{equation*}
\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^{2}}}\right)=\operatorname{tr}(E) \geqslant N(\operatorname{det} E)^{\frac{1}{N}}=N k_{0}^{\frac{1}{N}} \tag{3.9}
\end{equation*}
$$

Integrating now (3.7) and making use of the divergence theorem, boundary condition (1.6) and inequality (3.9), we are led to (1.10).

Obviously, the equality sign holds in (1.9) and (1.10) when $P(\mathbf{x}, 1) \equiv$ const. on $\bar{\Omega}$. In such a case, the boundary condition $u(\mathbf{x})=0$ on $\partial \Omega$ implies that $|\nabla u|^{2} \equiv$ const. on $\partial \Omega$. Therefore, according to a Serrin-type symmetry result (see L. Silvestre and B. Sirakov [14]), $\Omega$ must be a ball, while $u(\mathbf{x})$ must be radial (see G.A. Philippin and A. Safoui [10]). In conclusion, the equality sign holds in (1.9) and (1.10) if and only if $\Omega$ is a ball and $S$ is an $N$-dimensional hemisphere. The proof of Theorem 1.2 is thus achieved.

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[^0]:    E-mail address: cenache23@yahoo.com.
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