Partial differential equations

Symmetry of components and Liouville theorems for noncooperative elliptic systems on the half-space

Symétrie des composantes et théorèmes de Liouville pour des systèmes elliptiques non coopératifs dans le demi-espace

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ABSTRACT

We study classical solutions of elliptic systems in the half-space and provide sufficient conditions for having symmetry (or proportionality) of components, i.e. \( u = K v \) with \( K > 0 \), which then reduces the system to the scalar case. Under a natural structure condition on the nonlinearities, we show that solutions with sublinear growth, hence in particular bounded solutions, are symmetric. Noncooperative, nonvariational systems as well as supercritical nonlinearities can be covered. We also give an instance of our proportionality results without growth restriction on the solutions. As a consequence, we obtain new Liouville-type theorems in the half-space, as well as a priori estimates and existence results for related Dirichlet problems. Our proofs are based on a maximum principle, on the properties of suitable half-spherical means, on a rigidity result for superharmonic functions and on nonexistence of solution for scalar inequalities on the half-space.

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RÉSUMÉ

Nous étudions les solutions classiques de systèmes elliptiques dans le demi-espace et donnons des conditions suffisantes assurant la symétrie (ou la proportionnalité) des composantes, i.e. \( u = K v \) avec \( K > 0 \), ce qui réduit alors le système au cas scalaire. Sous une condition naturelle de structure sur les non-linéarités, nous montrons que les solutions à croissance sous-linéaire, donc en particulier les solutions bornées, sont symétriques. Ce résultat couvre le cas de systèmes non coopératifs, non variationnels et éventuellement sur-critiques. Nous obtenons aussi des résultats de proportionnalité sans hypothèse de croissance sur les solutions. Comme conséquence, nous obtenons de nouveaux théorèmes de type Liouville dans le demi-espace, ainsi que des estimations a priori et des résultats d'existence pour des problèmes de Dirichlet associés. Nos preuves reposent sur un principe du maximum, sur les propriétés de moyennes semi-sphériques, sur un résultat de rigidité pour les fonctions surharmoniques et sur la nonexistence de solution pour des inéquations scalaires dans le demi-espace.

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Nous étudions des systèmes elliptiques semi-linéaires dans le demi-espace $H_n$, $n \geq 1$, du type:
\[
\begin{cases}
-\Delta u = f(u, v) \quad &\text{sur } H_n, \\
-\Delta v = g(u, v) \quad &\text{sur } H_n, \\
u = v = 0 \quad &\text{sur } \partial H_n
\end{cases}
\]
et donnons des conditions suffisantes pour avoir la proportionnalité des composantes, i.e. $u = Kv$, ce qui permet alors de se ramener au cas scalaire et de classifier les solutions ou obtenir de nouveaux résultats de type Liouville. La condition de structure (2) apparaît naturellement et permet d'obtenir le théorème général suivant, valable pour les solutions à croissance sous-linéaire, donc pour les solutions bornées (ce qui est suffisant pour l’application de la méthode de renormalisation de Gidas et Spruck [7]). Notons que nos résultats peuvent aussi s’appliquer à des non-linéarités sur-critiques.

**Théorème 0.1.** Soit $(u, v) \in C^2(\overline{H_n})^2$ une solution de (1) où $f$ et $g$ satisfont (2). Si $u$ et $v$ sont à croissance sous-linéaire, i.e. $u(x) = o(|x|)$ et $v(x) = o(|x|)$ quand $x \to \infty$, alors $u = Kv$.

Ce théorème est optimal, comme le montre l’exemple $u = x_n, v = 0$, $f(u, v) = g(u, v) = uv, K = 1$. Il s’applique au système (4) comme au système non coopératif et non variationnel (5) et permet, par exemple, d’obtenir le théorème de Liouville suivant (grâce aux résultats de [7,8]) :

**Théorème 0.2.** On note $\sigma = p + q + r$.

(i) La seule solution bornée de (4) ou (5) est la solution triviale.

(ii) Si $\sigma \leq (n + 2)/(n - 2)_+$, alors la seule solution à croissance sous-linéaire est la solution triviale.

On peut également obtenir des résultats de proportionnalité sans faire d’hypothèse de croissance sur les solutions. Par exemple, appelant semi triviale une solution telle que $u = 0$ ou $v = 0$, on obtient, à l’aide d’un résultat de rigidité pour les fonctions surharmoniques dans le demi-espace, le théorème suivant :

**Théorème 0.3.** Soit $(u, v) \in C^2(\overline{H_n})^2$ une solution positive ou nulle du système (1), où l’on suppose que $f$ et $g$ satisfont la condition (2) et que :
\[
f(u, v) \geq cu^p v^q + r \quad \text{et} \quad g(u, v) \geq cu^r v^p
\]
où $c > 0$ et $p, q, r > 0$. On note $\sigma = p + q + r$.

(i) On $u \leq K v$ ou $u \geq K v$.

(ii) Si $\eta \leq (n + 1 + p + q)/(n - 1)$ et $p \leq (n + 1 + r + q)/(n - 1)$, alors $u = K v$ ou $(u, v)$ est semi triviale.

(iii) Si on suppose la condition plus forte $\eta \leq (n + 1)/(n - 1)$, alors $(u, v)$ est semi triviale.

Comme application, nous obtenons des résultats d’existence et d’estimation a priori pour le problème de Dirichlet (6). Celui-ci comprend comme cas particuliers des modèles d’interaction symbiotique d’espèces biologiques (de type Lotka–Volterra), de condensats de Bose–Einstein et de réactions chimiques.

1. Introduction

In order to show existence of a classical solution for semilinear elliptic systems of the form:
\[
\begin{cases}
-\Delta u = f(x, u, v), \quad &x \in \Omega, \\
-\Delta v = g(x, u, v), \quad &x \in \Omega
\end{cases}
\]
in a bounded regular domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with Dirichlet boundary conditions, a well-known method is to first show an a priori estimate for all solutions and then to apply a topological method via degree theory (see [1] for instance). To prove the former, we can use the rescaling (or blow-up) method introduced in [7] (see, e.g., [2] for the case of systems), which requires the knowledge of Liouville-type theorems for bounded solutions in the case of the whole space $\mathbb{R}^n$ and the half-space:
\[
H_n = \{x \in \mathbb{R}^n, \ x_n > 0\}.
\]
To get these nonexistence results, most methods seem to use either moving planes (or spheres) and Kelvin transform (and hence require a rather restrictive assumption of cooperativity of the system) or Pohozaev-type identity (and hence require some variational structure).
Another possible method is, under the following natural condition:

\[ [f(u, v) - Kg(u, v)][u - Kv] \leq 0, \quad (u, v) \in \mathbb{R}^2, \text{ for some constant } K > 0 \]  

(2)

to show the proportionality of the components, i.e. \( u = Kv \), so as to reduce the system to a single equation. This then allows us to get new classification and nonexistence results by using known results for the scalar case. It is important to note that, since the rescaling method only "sees the highest order terms", only the latter have to satisfy condition (2).

This condition (2) is natural a priori, since in a bounded domain, it would imply proportionality just by considering \( w = u - Kv \) and integrating by parts, and a posteriori because of the generality of Theorem 2.1 in the half-space below. We would like to add that, more heuristically, if we consider the parabolic system associated with (2), the new form of systems (4) and (5) was considered in [13] and, for system (5), significantly improved results are given in [12].

Indeed, system (4) satisfies condition (2) with \( u = K v \) on \( \partial \Omega \) and \( K \) is the trivial one. Another clue to the attractivity of the diagonal is that condition (2) means that the vector field \((f, g)\) of the underlying differential system is pointing toward the latter.

This method has been employed successfully in [13] for general systems in the whole space (see also [10] for earlier use of this method for a particular system). In this Note, we focus on the case of the half-space \( \mathbb{H}_n \). Whereas a central ingredient in [13] was the use of spherical means, a key tool will here be the following half-spherical means:

\[
[u](R) = \frac{1}{R^2[S^+_R]^n} \int u(x)\,d\sigma_R(x), \quad R > 0,
\]

where \( S^+_R = \{ x \in H_n, |x| = R \} \). \( \sigma_R \) is Lebesgue's measure on \( S^+_R \) and \( |S^+_R| = \sigma_R(S^+_R) \). Complete proofs as well as further results obtained in collaboration with B. Sirakov will be provided in the forthcoming article [12].

2. Main results

The first result is a rather general theorem concerning classical solutions \( (u, v) \) with sublinear growth, i.e. such that \( u(x) = o(|x|) \) and \( v(x) = o(|x|) \) as \( x \to \infty \). This case in particular covers the case of bounded solutions, sufficient for the rescaling method.

**Theorem 2.1.** Assume that \( f \) and \( g \) satisfy (2). Let \( (u, v) \in C^2(\mathbb{H}_n)^2 \) be a solution of the system:

\[
-\Delta u = f(u, v) \quad \text{and} \quad -\Delta v = g(u, v) \quad \text{on } \mathbb{H}_n.
\]

(3)

If \( u = K v \) on \( \partial \mathbb{H}_n \) and \( u \) and \( v \) have sublinear growth, then \( u = K v \).

**Remark 2.** (a) This theorem is optimal since if one of the components has linear growth, the result is not valid anymore, as shown by the counterexample \( u = x_o, v = 0, f(u, v) = g(u, v) = uv, K = 1 \).

(b) Note that no assumption is made on the sign of the solutions or of the nonlinearities \( f \) and \( g \).

This theorem in particular applies to the nonnegative solutions of the following two systems:

\[
\begin{cases}
-\Delta u = u^qv^{p+q} & \text{on } \mathbb{H}_n \\
-\Delta v = u^rv^p & \text{on } \mathbb{H}_n \\
u = v = 0 & \text{on } \partial \mathbb{H}_n
\end{cases}
\]

(4)

\[
\begin{cases}
-\Delta u = u^qv^{p+q} & \text{on } \mathbb{H}_n \\
-\Delta v = v^{p}[av^q - cu^q] & \text{on } \mathbb{H}_n \\
u = v = 0 & \text{on } \partial \mathbb{H}_n
\end{cases}
\]

(5)

Indeed, system (4) satisfies condition (2) with \( K = 1 \) and it can be shown (see [12]) that (5) satisfies (2) for some (unique) \( K > 0 \) (\( K \) being equal to 1 if and only if \( a + d = b + c \)). Hence, thanks to the scalar nonexistence results in [7,8], we can deduce the following Liouville-type result.

**Theorem 2.2.** Denote \( \sigma = p + q + r \).

(i) The only nonnegative bounded solution of (4) or (5) is the trivial one.

(ii) If \( \sigma \leq (n+2)/(n-2)_+ \), then the only solution with sublinear growth is the trivial one.

**Remark 2.** Note that system (5) is not cooperative (and generally non-variational) for \( p > 0 \). The whole space case for systems (4) and (5) was considered in [13] and, for system (5), significantly improved results are given in [12].
Concerning solutions without any growth assumption, we give an instance of our results, relying on a rigidity result for superharmonic functions in the half-space. We will say that a solution \((u, v)\) is **semitrivial** if \(u = 0\) or \(v = 0\) and that it is **positive** if \(u > 0\) and \(v > 0\).

**Theorem 2.3.** Let \((u, v) \in C^2(\Omega_n)^2\) be a nonnegative solution of system (3) with boundary conditions \(u = v = 0\) on \(\partial H_n\). We assume that \(f, g\) satisfy condition (2) and that:

\[
 f(u, v) \geq cu^{p+q} \quad \text{and} \quad g(u, v) \geq cu^{q+p} \quad \text{for all } u, v \geq 0,
\]

for some \(c > 0\), where \(p, q, r > 0\). We denote \(\sigma = p + q + r\).

1. We have \(u \leq K_v\) or \(u \geq K_v\).
2. If \(r \leq (n + 1 + p + q)/(n - 1)\) and \(p \leq (n + 1 + r + q)/(n - 1)\), then \(u = K_v\) or \((u, v)\) is semitrivial.
3. If we assume the stronger condition \(\sigma \leq (n + 1)/(n - 1)\), then \((u, v)\) is semitrivial.

**Remark 3.** (a) The previous theorem allows us to address supercritical nonlinearities, as can be seen by taking \(q\) large enough in system (4) for instance.

(b) Considering system (4), under conditions in (ii), by a well-known Liouville-type result of [7], we can deduce that if \(\sigma \leq (n + 2)/(n - 2)_+\), then the system has no positive solution.

As an application, let us now consider the Dirichlet problem:

\[
\begin{cases}
-\Delta u = u^p [a(x)v^q - c(x)u^q] + \lambda(x)u, & x \in \Omega, \\
-\Delta v = v^p [b(x)u^q - d(x)v^q] + \mu(x)v, & x \in \Omega, \\
u = v = 0, & x \in \partial \Omega,
\end{cases}
\]

where \(\Omega\) is a smoothly bounded domain of \(\mathbb{R}^n\), and that \(a, b, c, d, \lambda, \mu \in C(\Omega)\) satisfy \(a, b > 0, c, d, \lambda, \mu \geq 0\) in \(\Omega\). Note that the special cases: (a): \(r = q = 1, p = 0\); (b): \(r = 1, p = 0, q = 2\); and (c): \(r = p = q = 1\) respectively correspond to a Lotka–Volterra-type system modeling the symbiotic interaction of biological species, to models of Bose–Einstein condensates, and to systems involved in models of chemical reaction. These systems were considered for instance in [10,4], [9,3], [5]. The following theorem extends or improves some of the results therein.

**Theorem 2.4.** Assume \(q \geq |p - r|, q + r \geq 1, 1 < p + q + r < (n + 2)/(n - 2)_+\) and

\[
\sup_{x \in \Omega} \frac{c(x)d(x)}{a(x)b(x)} < 1.
\]

(i) There exists \(M > 0\) such that any positive classical solution \((u, v)\) of (6) satisfies \(\|u\|_{\infty}, \|v\|_{\infty} \leq M\).

(ii) Assume in addition that \(a, b, c, d, \lambda, \mu\) are Hölder continuous and that \(\lambda, \mu < \lambda_1\) on \(\Omega\).

Then there exists at least a positive classical solution of (6).

3. Sketch of proofs

**Proof of Theorem 2.1.** We denote \(w = u - K_v\). Since \(u\) and \(v\) have sublinear growth, then \(|w|\) also. Hence \(|w_+|_R \rightarrow 0\) on \(\partial H_n\) and \((w_+)_R \rightarrow 0\). So, thanks to condition (2), we can apply the following key lemma to \(w\) and to \(-w\) to get \(w = 0\). □

**Lemma 3.1.** Let \(w \in C^2(\Omega_n)\) be such that \(w \leq 0\) on \(\partial H_n\) and \(\Delta w \geq 0\) on the set \(\{w > 0\}\). If we assume:

\[
\lim_{R \to \infty} \inf \{w_+ \}(R) = 0
\]

then \(w \leq 0\).

**Proof.** It follows by applying to \((a\) regularized version of\() w_+\) the formula:

\[
\frac{d}{dR}[z](R) = \frac{1}{R^2}|\mathcal{S}_R^2| \left[ \int_{B_R^+} \Delta z_R dx - \int_{B_R^+} z(y) dy \right], \quad R > 0, \quad z \in C^2(\Omega_n),
\]

which can be obtained by direct computation. □
For the proof of Theorem 2.3, we will need the following two lemmas. The first lemma is concerned with half-spherical means, and its second assertion gives a rigidity result for superharmonic functions on $H_n$.

**Lemma 3.2.** Let $u \in C^2_{\lambda}(\partial H_n)$, with $u \geq 0$. If $u$ is superharmonic on $H_n$, then $|u|$ is nonincreasing and $\lim_{R \to \infty} |u|(R) = \lambda \geq 0$. Moreover, $u(x) \geq \frac{\lambda}{|x|^n} x_n$ for all $x \in H_n$. (Note that $|x|^n$ is a constant.)

**Proof.** The first assertion follows from (8). The second assertion can be shown by representing the solutions via Poisson kernels on half-spheres and using the scaling properties of the kernels. Alternatively, it can be shown by applying to the function $w = \frac{1}{|x|^n} x_n - u$ a special kind of maximum principle, namely the corollary of Theorem 1, p. 341 in [6].

**Lemma 3.3.** Assume $r, s \geq 0$ and $c > 0$. Let $u \in C^2_{\lambda}(\partial H_n)$ be a nonnegative solution of $-\Delta u \geq c x_n^+ u^r$ on $H_n$. If $r \leq (n+1+s)/(n-1)$, then $u = 0$.

**Proof.** It is based on rescaled test functions, in a similar way as in [11, Theorem 10.1, p. 36], which concerns the inequality $-\Delta u \geq |x|^\sigma u^r$ in a half-space.

**Proof of Theorem 2.3.** (i) The functions $f$ and $g$ are nonnegative, so $u$ and $v$ are superharmonic. Then by Lemma 3.2,

$$\lim_{R \to \infty} |u|(R) = \lambda \geq 0 \quad \text{and} \quad \lim_{R \to \infty} |v|(R) = \mu \geq 0.$$

- Assume $\lambda > 0$ and $\mu > 0$. Since $u = 0$ and $v = 0$ on $\partial H_n$, by Lemma 3.2(i), we have, for some $c > 0$,

$$u(x) \geq c x_n \quad \text{and} \quad v(x) \geq c x_n, \quad x \in H_n.$$

Hence, $-\Delta u \geq c x_n^\sigma$, so by Lemma 3.3 with $r = 0$, we have a contradiction.

- Assume $\lambda = 0$. Then, setting $w = u - K v$, we have $w_+ \leq u$ since $u, v \geq 0$ and then $\lim_{R \to \infty} [w_+](R) = 0$. By Lemma 3.1, this implies $w \leq 0$, i.e. $u \leq K v$.

- If $\mu = 0$, similarly, we get $u \leq K v$.

(ii) Using the notation of (i), if $\lambda > 0$, then $-\Delta v \geq c x_n^{r+q} v^p$. Since $p \leq (n + 1 + r + q)/(n - 1)$, we have $v = 0$ by Lemma 3.3. Similarly, if $\mu > 0$, then $u = 0$. If $\lambda = \mu = 0$, then, by the proof of (i), we get $u = K v$.

(iii) If $\sigma \leq (n+1)/(n-1)$, since $-\Delta u \geq c u^\sigma$ or $-\Delta v \geq c v^\sigma$ due to assertion (i), we deduce from Lemma 3.3 that $u = 0$ or $v = 0$.

**Proof of Theorem 2.4.** Assertion (i) is obtained by using the rescaling method of [13], relying on Liouville-type theorems in a half-space (Theorem 2.2) and in the whole space [13,12]. Some special care is needed in order to rule out semitrivial limits (i.e., solutions of the form $(0, C)$ or $(C, 0)$) in the rescaling procedure (this is based on an eigenfunction argument on large balls and uses the assumption $r \leq 1$). As for assertion (ii), it follows from classical topological degree arguments.

**References**


