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Partial differential equations

Local well-posedness of a nonlinear KdV-type equation

Existence locale pour une équation non linéaire de type KdV

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ABSTRACT

In this paper, a generalized nonlinear KdV equation with time- and space-dependent coefficients is considered. We show that the control of the dispersive and "diffusion" terms is possible if we use an adequate weight function determined with respect to the dispersive and "diffusion" coefficients to define the energy. We use the dispersive properties of the equation to prove the existence and uniqueness of solutions.

RÉSUMÉ

Dans cette note, on considère une équation de KdV généralisée avec coefficients variables en temps et en espace. On montre que les termes de « diffusion » et de dispersion peuvent être contrôlés en utilisant une fonction poids, déterminée en fonction des coefficients de « diffusion » et de dispersion, appropriée pour définir l'énergie ; puis, en utilisant la propriété de dispersion de l'équation, on montre un résultat d'existence et d'unicité des solutions.

Version française abrégée

On montre, dans cette note, l'existence et l'unicité des solutions du problème de Cauchy pour une équation de KdV généralisée (1) à coefficients variables en temps et en espace. Cette équation non linéaire est beaucoup utilisée pour décrire des phénomènes physiques qui restent difficiles à comprendre. Citons, par exemple, le phénomène de propagation des ondes faiblement non linéaires, ainsi que le phénomène de déferlement des vagues, tous deux en régime d'eau peu profonde et sur un fond de hauteur variable (voir aussi les commentaires dans [6,5]). Le fait de trouver des solutions au problème (1) contribue, d'une manière significative, à l'étude de la dynamique de ces phénomènes ; en fait, cela permet de s'assurer que le modèle considéré est mathématiquement bien posé. À notre connaissance il n'y a pas de résultat d'existence et d'unicité de solutions régulières de cette équation. On prouve ici, sous certaines conditions sur le coefficient de dispersion $a_3(t,x)$, l'existence locale en temps et l'unicité des solutions via un schéma itératif classique de Picard.

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Notre résultat principal concernant le problème (1) est donné dans le théorème suivant :

Théorème 0.1. *Soient $s > \frac{3}{2}$, $p \in \mathbb{N}^*$ et $f \in C([0,T]; H^s(\mathbb{R}))$. On suppose que :*

- a_0, a_1 *et* $a_2 \in C([0,T]; H^s(\mathbb{R}))$,
- $a_3 \in C([0,T]; L^\infty(\mathbb{R}))$, $\partial_x a_3 \in C([0,T]; H^{s+1}(\mathbb{R}))$ *avec* $\partial_t a_3 \in L^\infty(0,T; L^\infty(\mathbb{R}))$.
- $b \in C([0,T]; H^{s+1}(\mathbb{R}))$.
- $F(t,x) := \int_0^x \frac{b}{a_3}\,\mathrm{d}y \in C([0,T]; L^\infty(\mathbb{R}))$ *avec* $\partial_t F \in L^\infty(0,T; L^\infty(\mathbb{R}))$.

On suppose, de plus, que le coefficient de dispersion $a_3(t,x)$ est non dégénéré, c'est-à-dire qu'il existe une constante $c_0 > 0$ telle que $c_0 \leqslant |a_3(t,x)| \forall (t,x) \in [0,T] \times \mathbb{R}$. Alors, pour tout $u^0 \in H^s(\mathbb{R})$, il existe un instant $T^\star > 0$ et une unique solution u de (1) dans $C([0,T^\star]; H^s(\mathbb{R}))$.

1. Presentation of the problem

In this paper, we study the Cauchy problem for the general nonlinear KdV-type equation:

$$\begin{cases} u_t + a_0 u_x + a_1 u + a_2 u^p u_x + b u_{xx} + a_3 u_{xxx} = f, & \text{for } (t,x) \in (0,T] \times \mathbb{R}, \\ u|_{t=0} = u^0, \end{cases} \tag{1}$$

where $u = u(t,x)$, from $[0,T] \times \mathbb{R}$ into $\mathbb{R}$, is the unknown of the problem, $b(t,x)$ and $a_i = a_i(t,x)$ are real-valued smooth given functions of $(t,x) \in [0,T] \times \mathbb{R}$, and $p \in \mathbb{N}^*$ is given. This nonlinear partial differential equation is widely used to describe complex phenomena in various fields of science, especially in fluid mechanics. For instance, this initial value problem models the propagation of weak nonlinear dispersive waves, and it also describes the breaking phenomenon of soliton-like waves, in a varied-depth shallow-water tunnel (as is pointed in [6,5]) for both cases. Looking for solutions of (1) plays an important and significant role in the study of the dynamics of those phenomena and, to our knowledge, the problem (1) has not yet been analyzed. We prove, in the present article, the local well-posedness of the initial value problem (1) by a standard Picard iterative scheme under a condition of non-degeneracy of the dispersive coefficient $a_3(t,x)$. Note that a linear form of this KdV-type problem is studied in a recent paper [2].

2. Notation

In the following, C_0 denotes any nonnegative constant whose exact expression is of no importance. The notation $a \lesssim b$ means that $a \leqslant C_0 b$. We designate by $C(\lambda_1, \lambda_2, \ldots)$ a nonnegative constant depending on the parameters $\lambda_1, \lambda_2, \ldots$ and whose dependence on the λ_j's is always assumed to be nondecreasing. For any constant r such that $1 \leqslant r \leqslant \infty$, $L^r = L^r(\mathbb{R})$ denotes the space of all Lebesgue-measurable functions f, equipped with its usual norm. We simply denote the norm $|\cdot|_{L^2}$ by $|\cdot|_2$. The real inner product of any functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R})$ is denoted by $(f_1, f_2) = \int_{\mathbb{R}} f_1(x) f_2(x)\,\mathrm{d}x$.

For integer m, we denote by $W^{m,\infty} = \{f, \partial_x^i f \in L^\infty(\mathbb{R}) \text{ for } i \leqslant m\}$ the classical Sobolev space endowed with its canonical norm.

For any real constant $s \geqslant 0$, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H^s} = |\Lambda^s f|_2 < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1 - \partial_x^2)^{1/2}$.

For all functions $u = u(t,x)$ and $v(t,x)$ defined on $[0,T] \times \mathbb{R}$ with $T > 0$, we denote the inner product in L^2, the L^2-norm, and the Sobolev norm in H^s, with respect to the spatial variable x, by $(u,v) = (u(t,\cdot), v(t,\cdot))$, $|u|_{L^2} = |u(t,\cdot)|_{L^2}$, and $|u|_{H^s} = |u(t,\cdot)|_{H^s}$, respectively.

We denote by $C^k([0,T); H^s(\mathbb{R}))$ the space of k-times continuously differentiable functions u with values in $H^s(\mathbb{R})$, with the norm: $\|u\|_{C^k([0,T);H^s(\mathbb{R}))} = \max_{0 \leqslant i \leqslant k} \sup_{t \in [0,T)} |\partial_x^i u(t,\cdot)|_{H^s} < \infty$. We denote by $L^\infty(0,T; L^\infty(\mathbb{R}))$ the space of L^∞ functions u with values in $L^\infty(\mathbb{R})$, with the norm: $\|u\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} = \operatorname{ess\,sup}_{t \in [0,T)} |u(t,\cdot)|_{L^\infty} < \infty$.

For any closed operator A defined on a Banach space X of functions, the commutator $[A,f]$ is defined by $[A,f]g = A(fg) - fA(g)$ with f, g and fg belonging to the domain of A.

3. The main result

We now study the local well-posedness of the initial value problem (1) in $H^s(\mathbb{R})$, for s large enough. More precisely, we prove the following theorem:

Theorem 3.1. *Let $s > \frac{3}{2}$, $p \in \mathbb{N}^*$ and $f \in C([0,T]; H^s(\mathbb{R}))$. We suppose that:*

- a_0, a_1 *and* $a_2 \in C([0,T]; H^s(\mathbb{R}))$,
- $a_3 \in C([0,T]; L^\infty(\mathbb{R}))$, $\partial_x a_3 \in C([0,T]; H^{s+1}(\mathbb{R}))$ *with* $\partial_t a_3 \in L^\infty(0,T; L^\infty(\mathbb{R}))$.
- $b \in C([0,T]; H^{s+1}(\mathbb{R}))$.

– *$F(t,x) := \int_0^x \frac{b}{a_3}\,\mathrm{d}y \in C([0,T]; L^\infty(\mathbb{R}))$ with $\partial_t F \in L^\infty(0,T; L^\infty(\mathbb{R}))$.*

Assume moreover that the dispersive coefficient $a_3(t,x)$ is non-degenerate, i.e., there is a positive constant $c_0 > 0$ such that $c_0 \leqslant |a_3(t,x)| \forall (t,x) \in [0,T] \times \mathbb{R}$. Then, for all $u^0 \in H^s(\mathbb{R})$, there exist a time $T^\star > 0$ and a unique solution u to (1) in $C([0,T^\star]; H^s(\mathbb{R}))$.

Remark 1. The regularity assumption on the dispersion coefficient a_3 with the non-degeneracy condition implies that a_3 keeps a constant sign on $[0,T] \times \mathbb{R}$.

Remark 2. There is no restriction on the sign of the coefficient b; this means that our result handles also the case of the anti-diffusive term, in which case this term is controlled by dispersion.

Proof. For any smooth enough v, we define the "linearized" operator:

$$\mathcal{L}(v,\partial) = \partial_t + a_0\partial_x + a_1 + a_2 v^p \partial_x + b\partial_x^2 + a_3\partial_x^3,$$

and the following initial value problem:

$$\begin{cases} \mathcal{L}(v,\partial)u = f, \\ u|_{t=0} = u^0. \end{cases} \tag{2}$$

Eq. (2) is a linear equation that can be solved in any time interval in which its coefficients are defined. We first establish some precise energy-type estimates of the solution. We then show that a control of the dispersive and diffusion terms ($a_3 u_{xxx}$ and $b u_{xx}$) is however possible under some conditions on the behavior of the quotient $\frac{b}{a_3}$ and if we use an adequate weight function to define the energy and use the dispersive properties of the equation. More precisely, inspired by [3], we define the "energy" norm,

$$E^s(u)^2 = \left|w\Lambda^s u\right|_2^2,$$

where the weight function w will be determined later ($s > 3/2$ is fixed here). For the moment, we just require that there exist two positive numbers w_1, w_2 such that for all (t,x) in $(0,T] \times \mathbb{R}$,

$$w_1 \leqslant w(t,x) \leqslant w_2,$$

so that $E^s(u)$ is uniformly equivalent to the standard H^s-norm. Differentiating $\frac{1}{2}\mathrm{e}^{-\lambda t}E^s(u)^2$ with respect to time, one gets, using (2):

$$\begin{aligned} \frac{1}{2}\mathrm{e}^{\lambda t}\partial_t\big(\mathrm{e}^{-\lambda t}E^s(u)^2\big) = {} & -\frac{\lambda}{2}E^s(u)^2 - \big(\big[\Lambda^s, a_0\big]\partial_x u, w^2\Lambda^s u\big) - \big(a_0\partial_x\Lambda^s u, w^2\Lambda^s u\big) \\ & - \big(w\Lambda^s(a_1 u), w\Lambda^s u\big) - \big(\big[\Lambda^s, a_2 v^p\big]\partial_x u, w^2\Lambda^s u\big) - \big(a_2 v^p\partial_x\Lambda^s u, w^2\Lambda^s u\big) \\ & - \big(\big[\Lambda^s, a_3\big]\partial_x^3 u, w^2\Lambda^s u\big) - \big(a_3\partial_x^3\Lambda^s u, w^2\Lambda^s u\big) - \big(\big[\Lambda^s, b\big]\partial_x^2 u, w^2\Lambda^s u\big) \\ & - \big(b\partial_x^2\Lambda^s u, w^2\Lambda^s u\big) + \big(\Lambda^s f, w^2\Lambda^s u\big) + \big(w_t\Lambda^s u, w\Lambda^s u\big). \end{aligned} \tag{3}$$

We now turn to estimating the different terms of the r.h.s. of the previous identity.

• Estimate of $([\Lambda^s, a_0]\partial_x u, w^2\Lambda^s u)$. We use here the Cauchy–Schwarz inequality and the following commutator estimate: for all F and U smooth enough, one has:

$$\left|\big[\Lambda^s, F\big]U\right|_2 \lesssim |\partial_x F|_{H^{s-1}}|U|_{H^{s-1}};$$

therefore,

$$\left|\big(\big[\Lambda^s, a_0\big]\partial_x u, w^2\Lambda^s u\big)\right| \leqslant \left|\big[\Lambda^s, a_0\big]\partial_x u\right|_2\left|w^2\Lambda^s u\right|_2 \leqslant C\big(|a_0|_{H^s}\big)E^s(u)^2.$$

• Estimate of $(a_0\partial_x\Lambda^s u, w^2\Lambda^s u)$. By integration by parts, we get:

$$\big(a_0\partial_x\Lambda^s u, w^2\Lambda^s u\big) = -\frac{1}{2}\big(\partial_x\big(a_0 w^2\big)\Lambda^s u, \Lambda^s u\big);$$

it is now easy to check that one gets:

$$\left|\big(a_0\partial_x\Lambda^s u, w^2\Lambda^s u\big)\right| \leqslant C\big(|a_0|_{W^{1,\infty}}, |w|_{W^{1,\infty}}\big)E^s(u)^2.$$

• Estimate of $(w\Lambda^s(a_1 u), w\Lambda^s u)$. By the Cauchy–Schwarz inequality, we have:

$$\left|\big(w\Lambda^s(a_1 u), w\Lambda^s u\big)\right| \leqslant C\big(|a_1|_{H^s}, |w|_{W^{1,\infty}}\big)E^s(u)^2.$$

• Estimate of $([\Lambda^s, a_2 v^p]\partial_x u, w^2\Lambda^s u)$. We may proceed as for the estimate of $([\Lambda^s, a_0]\partial_x u, w^2\Lambda^s u)$ and obtain:

$$\left|\left(\left[\Lambda^s, a_2 v^p\right]\partial_x u, w^2\Lambda^s u\right)\right| \leqslant C\left(p, |a_2|_{H^s}, |v|_{H^s}\right)E^s(u)^2.$$

• Estimate of $(a_2 v^p \partial_x \Lambda^s u, w^2\Lambda^s u)$. It is clear, by a simple integration by parts, that:

$$\left|\left(a_2 v^p \partial_x \Lambda^s u, w^2\Lambda^s u\right)\right| \leqslant C\left(p, |a_2|_{W^{1,\infty}}, |v|_{W^{1,\infty}}, |w|_{W^{1,\infty}}\right)E^s(u)^2. \tag{4}$$

• Estimate of $([\Lambda^s, a_3]\partial_x^3 u, w^2\Lambda^s u) + (a_3\partial_x^3\Lambda^s u, w^2\Lambda^s u) + ([\Lambda^s, b]\partial_x^2 u, w^2\Lambda^s u) + (b\partial_x^2\Lambda^s u, w^2\Lambda^s u)$. Let us now focus on the seventh and eighth terms of the right-hand side of the identity (3). In order to get an adequate estimate of the seventh term, we have to explicitly write the commutator $[\Lambda^s, a_3]$:

$$\left[\Lambda^s, a_3\right]\partial_x^3 u = \left\{\Lambda^s, a_3\right\}_2 \partial_x^3 u + Q_1\partial_x^3 u,$$

where $\{\cdot,\cdot\}_2$ stands for the second-order Poisson brackets,

$$\left\{\Lambda^s, a_3\right\}_2 = -s\partial_x(a_3)\Lambda^{s-2}\partial_x + \frac{1}{2}\left[s\partial_x^2(a_3)\Lambda^{s-2} - s(s-2)\partial_x^2(a_3)\Lambda^{s-4}\partial_x^2\right]$$

and Q_1 is an operator of order $s-3$ that can be controlled by the general commutator estimates (see Theorem 6 in [4]). We thus get:

$$\left|\left(Q_1\partial_x^3 u, w^2\Lambda^s u\right)\right| \leqslant C\left(|\partial_x a_3|_{H^{s+1}}\right)E^s(u)^2.$$

We now use the identity $\Lambda^2 = 1 - \partial_x^2$ and the fact that $H^1(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$ to get, as in (4),

$$\left|\left(\left[s\partial_x^2(a_3)\Lambda^{s-2} - s(s-2)\partial_x^2(a_3)\Lambda^{s-4}\partial_x^2\right]\partial_x^3 u, w^2\Lambda^s u\right)\right| \leqslant C\left(s, |\partial_x a_3|_{H^{s+1}}, |w|_{W^{1,\infty}}\right)E^s(u)^2.$$

This leads to the expression:

$$\left(\left[\Lambda^s, a_3\right]\partial_x^3 u, w^2\Lambda^s u\right) = s\left(\partial_x(a_3)\Lambda^s\partial_x^2 u, w^2\Lambda^s u\right) + Q_2,$$

where $|Q_2| \leqslant C(s, |w|_{W^{1,\infty}}, |\partial_x a_3|_{H^{s+1}})E^s(u)^2$. Remarking now that, by integration by parts,

$$s\left(\partial_x(a_3)\Lambda^s\partial_x^2 u, w^2\Lambda^s u\right) = -s\left(\partial_x\left(\partial_x(a_3)w^2\right)\Lambda^s\partial_x u, \Lambda^s u\right) - s\left(\partial_x(a_3)w^2, \left(\Lambda^s\partial_x u\right)^2\right), \tag{5}$$

we may control the eighth term in a similar way, after calculating:

$$\left(a_3\partial_x^3\Lambda^s u, w^2\Lambda^s u\right) = -\frac{1}{2}\left(\partial_x^3\left(a_3 w^2\right)\Lambda^s u, \Lambda^s u\right) - \frac{3}{2}\left(\partial_x^2\left(w^2 a_3\right)\Lambda^s\partial_x u, \Lambda^s u\right) - \frac{3}{2}\left(\partial_x\left(w^2 a_3\right)\Lambda^s u, \Lambda^s\partial_x^2 u\right).$$

In the same way, as in (5), we obtain:

$$-\frac{3}{2}\left(\partial_x\left(w^2 a_3\right)\Lambda^s u, \Lambda^s\partial_x^2 u\right) = \frac{3}{2}\left(\partial_x^2\left(a_3 w^2\right)\Lambda^s u, \Lambda^s\partial_x u\right) + \frac{3}{2}\left(\partial_x\left(a_3 w^2\right), \left(\Lambda^s\partial_x u\right)^2\right).$$

We now focus on the ninth and tenth terms of the right-hand side of the identity (3) by using the same way as above, with the first-order Poisson brackets $\{\Lambda^s, b\}_1 = -s\partial_x(b)\Lambda^{s-2}\partial_x$, we get:

$$\left(\left[\Lambda^s, b\right]\partial_x^2 u, w^2\Lambda^s u\right) = s\left(\partial_x(b)\Lambda^s\partial_x u, w^2\Lambda^s u\right) + Q_4,$$

where $|Q_4| \leqslant C(s, |w|_{W^{2,\infty}}, |b|_{H^{s+1}})E^s(u)^2$. Remarking now that, by integration by parts,

$$s\left(\partial_x(b)\Lambda^s\partial_x u, w^2\Lambda^s u\right) = -\frac{s}{2}\left(\partial_x\left(\partial_x(b)w^2\right)\Lambda^s u, \Lambda^s u\right) \tag{6}$$

then

$$\left|\left(\left[\Lambda^s, b\right]\partial_x^2 u, w^2\Lambda^s u\right)\right| \leqslant C\left(s, |w|_{W^{2,\infty}}, |b|_{H^{s+1}}\right)E^s(u)^2$$

we may control the tenth term in a similar way:

$$\left(b\partial_x^2\Lambda^s u, w^2\Lambda^s u\right) = -\left(bw^2, \left(\partial_x\Lambda^s u\right)^2\right) + Q_5$$

where $|Q_5| \leqslant C(s, |w|_{W^{1,\infty}}, |\partial_x b|_{L^\infty})E^s(u)^2$. We now choose w such that:

$$-s\left(\partial_x(a_3)w^2, \left(\Lambda^s\partial_x u\right)^2\right) + \frac{3}{2}\left(\partial_x\left(a_3 w^2\right), \left(\Lambda^s\partial_x u\right)^2\right) + \left(bw^2, \left(\partial_x\Lambda^s u\right)^2\right) = 0; \tag{7}$$

therefore, if we take $w = |a_3|^{(\frac{2s-3}{6})} \exp(-\frac{1}{3}\int_0^x \frac{b}{a_3}\,dy)$, we easily obtain (7). We can check by using the assumptions in Theorem 3.1, that w is in $C([0,T]; W^{3,\infty})$ and $\partial_t w \in L^\infty(0,T; L^\infty)$. Finally, one has:

$$\begin{aligned}&\big([\Lambda^s, a_3]\partial_x^3 u, w^2\Lambda^s u\big) + \big(a_3\partial_x^3\Lambda^s u, w^2\Lambda^s u\big)\\ &\quad = Q_2 + s\big(\partial_x\big(\partial_x(a_3)w^2\big)\Lambda^s\partial_x u, \Lambda^s u\big) - \frac{1}{2}\big(\partial_x^3\big(a_3 w^2\big)\Lambda^s u, \Lambda^s u\big)\\ &\qquad - \frac{3}{2}\big(\partial_x^2\big(a_3 w^2\big)\Lambda^s\partial_x u, \Lambda^s u\big) + \frac{3}{2}\big(\partial_x^2\big(a_3 w^2\big)\Lambda^s\partial_x u, \Lambda^s u\big);\end{aligned}$$

therefore

$$\begin{aligned}&\big|\big([\Lambda^s, a_3]\partial_x^3 u, w^2\Lambda^s u\big) + \big(a_3\partial_x^3\Lambda^s u, w^2\Lambda^s u\big) + \big([\Lambda^s, b]\partial_x^2 u, w^2\Lambda^s u\big) + \big(b\partial_x^2\Lambda^s u, w^2\Lambda^s u\big)\big|\\ &\quad \leqslant C\big(s, |b|_{H^{s+1}}, |a_3|_{L^\infty}, |\partial_x a_3|_{H^{s+1}}, |w|_{W^{3,\infty}}\big)E^s(u)^2.\end{aligned}$$

- Estimate of $(w_t\Lambda^s u, w\Lambda^s u)$. By use of the Cauchy–Schwarz inequality, we obtain:

$$\big|\big(w_t\Lambda^s u, w\Lambda^s u\big)\big| \leqslant C\big(|w_t|_{L^\infty}, |w|_{L^\infty}\big)E^s(u)^2.$$

Gathering the information provided by the above estimates, since one has:

$$\big|\big(\Lambda^s f, w^2\Lambda^s u\big)\big| \leqslant E^s(f)E^s(u),$$

we obtain:

$$e^{\lambda t}\partial_t\big(e^{-\lambda t}E^s(u)^2\big) \leqslant \big(C\big(E^s(v)\big) - \lambda\big)E^s(u)^2 + 2E^s(f)E^s(u).$$

Taking $\lambda = \lambda_T$ large enough (how large depends on $\sup_{t\in[0,T]} C(E^s(v(t)))$ for the first term of the right-hand side of the above inequality to be negative for all $t \in [0,T]$, we deduce:

$$E^s\big(u(t)\big) \leqslant e^{\lambda_T t}E^s\big(u^0\big) + 2\int_0^t e^{\lambda_T(t-t')}E^s\big(f(t')\big)\,dt'.$$

Thanks to this energy estimate, we classically conclude (see, e.g., [1]) to the existence of a time:

$$T^* = T^*\big(E^s\big(u^0\big)\big) > 0,$$

and of a unique solution $u \in C([0,T^*]; H^s(\mathbb{R})) \cap C^1([0,T^*]; H^{s-3}(\mathbb{R}))$ to (1) as the limit of the iterative scheme:

$$u_0 = u^0, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad \begin{cases}\mathcal{L}(u^n, \partial)u^{n+1} = f,\\ u^{n+1}_{|t=0} = u^0.\end{cases} \qquad \square$$

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