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# The projective plane, $J$ -holomorphic curves and Desargues' theorem



*Plan projectif, courbes  $J$ -holomorphes et théorème de Desargues*

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## ABSTRACT

By a theorem of Gromov, for an almost complex structure  $J$  on  $\mathbb{C}P^2$  tamed by the standard symplectic structure, the  $J$ -holomorphic curves representing the positive generator of homology form a projective plane. We show that this satisfies the Theorem of Desargues if and only if  $J$  is isomorphic to the standard complex structure. This answers a question of Ghys.

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## R É S U M É

D'après un théorème dû à Gromov, pour toute structure presque complexe  $J$  sur  $\mathbb{C}P^2$  qui est compatible avec la structure symplectique standard, les courbes  $J$ -holomorphes dans la classe du générateur positif de  $H_2(\mathbb{C}P^2)$  correspondent à la collection des droites d'un plan projectif. Nous démontrons que ce plan projectif est arguésien si et seulement si  $J$  est isomorphe à la structure complexe standard, répondant ainsi à une question posée par Ghys.

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## 1. Introduction

Consider an almost complex structure  $J$  on  $\mathbb{C}P^2$ , which is tamed by the standard symplectic form  $\omega$ , i.e., such that  $\omega(v, Jv) > 0$  for all non-zero tangent vectors  $v$ . Then, by a theorem of Gromov [2], we can associate a projective plane (we recall definitions below) with  $J$ , with points the points of  $\mathbb{C}P^2$  and lines  $J$ -holomorphic curves  $\Sigma$  such that the corresponding homology class  $[\Sigma] \in H_2(\mathbb{C}P^2, \mathbb{Z})$  is the positive generator (i.e., the generator on which the integral of  $\omega$  is positive). Ghys asked whether the Theorem of Desargues for such a projective plane implies that  $J$  is integrable, or equivalently standard. We show that this is indeed the case.

**Theorem 1.1.** *Suppose that the geometry associated with  $J$  satisfies the theorem of Desargues; then, there is a diffeomorphism from  $\mathbb{C}P^2$  to itself that maps the standard complex structure to  $J$ .*

Our proof is a simple extension of classical results, for which we refer to Emil Artin's *Geometric Algebra* [1], where it is shown that such a projective plane is the projective plane over a division ring  $k$ . We refine the arguments to show that in

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our case  $k$  is a topological division ring homeomorphic to  $\mathbb{C}$ . By a theorem of Pontryagin, it follows that  $k = \mathbb{C}$ . We deduce the main result.

## 2. Affine and projective planes

An *affine plane* consists of collections of *points* and *lines*, with lines being sets of points. We assume that these satisfy certain axioms.

**Axiom** (*Axiom 1*). Through any pair of points  $P$  and  $Q$  there is a unique line  $PQ$ .

Hence, two distinct lines intersect in 0 or 1 points. We define lines to be parallel if they are disjoint or equal. Our second axiom is the so-called parallel postulate.

**Axiom** (*Axiom 2 (affine version)*). Given a line  $l$  and a point  $P$  not on  $l$ , there is a unique line through  $P$  parallel to  $l$ .

As a consequence, we note that being parallel is an equivalence relation. Symmetry and reflexivity are obvious. Transitivity is as below.

**Proposition 2.1.** *Suppose  $l'$  and  $l''$  are both parallel to  $l$ . Then  $l'$  is parallel to  $l''$ .*

**Proof.** Suppose not, then  $l'$  and  $l''$  intersect in a point  $P$ . Assuming that  $l$ ,  $l'$  and  $l''$  are all distinct, the lines  $l'$  and  $l''$  both pass through  $P$  parallel and are parallel to  $l$ , contradicting Axiom 2. The cases when two of the lines coincide are elementary.  $\square$

We shall call an equivalence class under this relation a *pencil of parallel lines*. We shall call the pencil of parallel lines containing a line  $l$  as the *direction* of  $l$ .

To ensure that our geometry is non-trivial, we also need another axiom.

**Axiom** (*Axiom 3*). There are three points  $A$ ,  $B$  and  $C$  not contained in a single line.

The basic example of an affine geometry is the set of pairs of points over a division ring  $k$ , with lines the set of solutions to non-degenerate linear equations over  $k$ .

We now turn to projective planes, which are completions of affine planes. These are also collections of points and lines satisfying appropriate axioms.

An affine plane gives rise to a projective plane. Namely, we add to an affine plane, points corresponding to the pencils of parallel lines. Each line  $l$  in the affine plane gives a line in the projective plane consisting of the points of  $l$  and the point at infinity corresponding to the direction of  $l$ . We also add a single line at infinity consisting of the new points (i.e., not in the affine plane).

The parallel postulate gets replaced by its projective counterpart.

**Axiom** (*Axiom 2 (projective version)*). Any pair of distinct lines intersect in a single point.

It is easy to see that Axioms 1 and 3 continue to hold for the projective plane corresponding to an affine plane. Conversely, given a projective plane satisfying Axioms 1 and 3 and the projective version of Axiom 2, we obtain an affine plane by deleting a line (which we declare to be the line to be at infinity) and all the points on it.

We now recall the Theorem of Desargues. This holds in the standard affine plane over a division ring  $k$ , but not necessarily in a general projective plane. We shall hence regard it as an additional axiom.

**Axiom** (*Theorem of Desargues*). Let  $O$  be a point in a projective plane,  $l_1$ ,  $l_2$  and  $l_3$  be distinct lines through  $O$  and  $A_i$ ,  $B_i$  be distinct points on  $l_i$ , for  $i = 1, 2, 3$ , none of which coincide with  $O$ . Then the points  $A_1A_2 \cap B_1B_2$ ,  $A_1A_3 \cap B_1B_3$  and  $A_2A_3 \cap B_2B_3$  are colinear.

We say that a projective plane is *Desarguesian* if it satisfies the Theorem of Desargues.

## 3. Projective plane from $J$ -holomorphic curves

Let  $J$  be an almost complex structure on  $\mathbb{C}P^2$  that is tamed by the standard symplectic structure, i.e., so that for every tangent vector  $V \in T_p\mathbb{C}P^2$ ,  $p \in \mathbb{C}P^2$ ,  $\omega(V, JV) > 0$ . Consider the set of  $J$ -holomorphic curves  $\Sigma$  in  $\mathbb{C}P^2$  such that  $[\Sigma]$  generates  $H_2(\mathbb{C}P^2, \mathbb{Z})$ . This will form the collection of lines in a projective plane. By a theorem of Gromov, this satisfies Axiom 1 for a projective plane.

**Theorem 3.1** (Gromov). *Given two points in  $\mathbb{C}P^2$ , there is a unique  $J$ -holomorphic curve  $\Sigma$  passing through  $P$  and  $Q$  so that  $[\Sigma] \in H_2(\mathbb{C}P^2, \mathbb{Z})$  is a generator.*

We remark that  $\omega$  evaluated on the class of  $\Sigma$  must be positive and  $\Sigma$  is topologically a sphere. Axiom 2 follows by the following elementary (and standard) proposition.

**Proposition 3.2.** *If  $\Sigma_1$  and  $\Sigma_2$  are distinct  $J$ -holomorphic curves homologous to the positive generator of  $H_2(\mathbb{C}P^2, \mathbb{Z})$ . Then  $\Sigma_1 \cap \Sigma_2$  is a single point and the intersection is transversal at this point.*

**Proof.** The algebraic intersection number of  $\Sigma_1$  and  $\Sigma_2$  is 1. As the curves are  $J$ -holomorphic they intersect in a finite number of points, each with positive multiplicity. Hence, by the strong positivity of intersections for holomorphic curves, they must intersect at exactly one point and this must have multiplicity one. In particular, the intersection point is transversal (as it has multiplicity 1).  $\square$

Thus, Axiom 2 holds. Axiom 3 follows as each line is a sphere and hence cannot be all of  $\mathbb{C}P^2$ . We consider henceforth such a projective plane.

#### 4. Basic continuity results

We next show that the basic geometric constructions we need are all continuous. We consider the topology on the space of lines given by the Hausdorff distance. Note that the Hausdorff topology coincides with the topology defined by Gromov in this context – namely, we view lines non-uniquely as holomorphic maps from  $\mathbb{C}P^1$  and consider the quotient of the  $C^0$ -topology on such maps. Our first lemma follows from standard constructions of moduli spaces of holomorphic maps.

**Lemma 4.1.** *The line  $PQ$  through distinct points  $P$  and  $Q$  varies continuously with  $P$  and  $Q$ .*

**Lemma 4.2.** *If  $l_1$  and  $l_2$  are distinct lines, then the point  $l_1 \cap l_2$  varies continuously as a function of  $l_1$  and  $l_2$ .*

**Proof.** Observe that if  $U$  is an open neighbourhood of  $l_1$  and  $l_2$ , then by compactness the distance between  $l_1 \setminus U$  and  $l_2 \setminus U$  is positive. Hence two lines  $l'_1$  and  $l'_2$ , sufficiently close to  $l_1$  and  $l_2$ , respectively, cannot intersect outside  $U$ . Hence their unique intersection point must lie in  $U$ . The result follows.  $\square$

Next, choose and fix a line, which we denote  $l_\infty$ , and consider the affine plane obtained by deleting this line.

**Lemma 4.3.** *Given a line  $l$  and a point  $P$  not on  $l$ , the line  $l'$  through  $P$  parallel to  $l$  varies continuously with  $l$  and  $P$ .*

**Proof.** The line  $l'$  is the unique line through  $P$  and the point  $l \cap l_\infty$ . Thus continuity follows from the previous lemmas.  $\square$

#### 5. Dilatations

We henceforth consider an affine plane obtained from the projective plane associated with an almost complex structure  $J$  as above.

**Definition 5.1.** *A dilatation of an affine plane is a map  $\varphi$  of points of the plane so that for distinct points  $P$  and  $Q$ , the point  $\varphi(Q)$  lies on the line through  $\varphi(P)$  parallel to the line  $PQ$ .*

Dilatations are determined by the images of two distinct points  $P$  and  $Q$ . We shall refine this to show that, in our case, dilatations are continuous functions depending continuously on the images of the points  $P$  and  $Q$ .

Namely, let  $R$ ,  $P'$  and  $Q'$  be points in the affine plane. Let  $\{P_n\}$ ,  $\{Q_n\}$ ,  $\{R_n\}$ ,  $\{P'_n\}$  and  $\{Q'_n\}$  be sequences of points converging to  $P$ ,  $Q$ ,  $R$ ,  $P'$  and  $Q'$  respectively.

**Lemma 5.2.** *A dilatation  $\varphi$  mapping  $P$  to  $P'$  and  $Q$  to  $Q'$ , if it exists, is unique. Further,  $\varphi$  is continuous, and if  $\{\varphi_n\}$  is a sequence of dilatations such that  $\varphi_n(P_n) = P'_n$  and  $\varphi_n(Q_n) = Q'_n$ , then  $\varphi_n(R_n)$  converges to  $\varphi(R)$ .*

**Proof.** Consider a sequence of points  $R_n$  converging to  $R$ . We shall show that  $\varphi_n(R_n)$  converges to  $\varphi(R)$ . All statements follow from this.

First, assume that  $R$  is not on the line  $PQ$ . Without loss of generality, we can then assume that  $R_n$  is not on the line  $P_nQ_n$  for all  $n$  (as this will be true for  $n$  sufficiently large). Observe that  $\varphi(R)$  lies on both the line through  $P' = \varphi(P)$  that is parallel to the line  $PR$  and the line through  $Q' = \varphi(Q)$  that is parallel to the line  $QR$ , and hence is the intersection

point of these lines. Similarly,  $\varphi_n(R_n)$  lies on both the line through  $P'_n = \varphi_n(P_n)$  that is parallel to the line  $P_nR_n$  and the line through  $Q'_n = \varphi_n(Q_n)$  that is parallel to the line  $Q_nR_n$ , and hence is the intersection point of these lines. By the lemmas of Section 4, we conclude that  $\varphi_n(R_n)$  converges to  $\varphi(R)$ .

If  $R$  is on the line  $PQ$ , we pick a point  $S$  not on the line  $PQ$  and deduce as above that  $\varphi_n(S)$  converges to  $\varphi(S)$ . We then apply the same argument with  $S$  in place of  $Q$ .  $\square$

In particular, the only dilatation that fixes at least two points is the identity. Every other dilatation has either no fixed points or a single fixed point. We call a dilatation with no fixed points a *translation*. The identity is also considered a translation.

## 6. The division ring $k$

We now recall the construction of the division ring  $k$  in the presence of Desargues' theorem. As the results in this section are classical (we follow [1]), we omit proofs. We shall see a more concrete description of  $k$ , including its topology, in the next section (but the correctness of that description depends on the results of this section).

We first consider translations.

**Lemma 6.1.** *If  $\tau \neq id$  is a translation, then lines through  $P$  and  $\tau(P)$ , for  $P$  in the affine plane, form a pencil of parallel lines.*

This pencil is called the *trace* of a translation. Henceforth assume that the Theorem of Desargues holds in the affine plane we have constructed.

**Lemma 6.2.** *Given a pair of points  $O$  and  $P$ , there is a unique translation  $\tau$  mapping  $O$  to  $P$ .*

**Lemma 6.3.** *The set of translations forms an Abelian group  $T$  under composition.*

A homomorphism  $\Phi$  of  $T$  (i.e., from  $T$  to itself) is said to be *trace-preserving* if  $\Phi(\tau)$  has the same trace as  $\tau$  for all translations  $\tau$ . We also allow  $\Phi(\tau)$  to be the identity.

The trace preserving homomorphisms form a ring  $k$  given by, for  $\Phi_1, \Phi_2 \in k$ ,

$$(\Phi_1 + \Phi_2)(\tau) := \Phi_1(\tau) \circ \Phi_2(\tau)$$

and

$$(\Phi_1 \cdot \Phi_2)(\tau) := \Phi_1(\Phi_2(\tau)).$$

Dilatations give trace-preserving homomorphisms of the Abelian group  $T$  by conjugation by the following lemma.

**Lemma 6.4.** *Suppose  $\varphi$  is a dilatation, then  $\tau \mapsto \varphi \circ \tau \circ \varphi^{-1}$  is a trace-preserving homomorphism of  $T$ .*

Fix a point  $O$  in the affine plane, a line  $l$  through  $O$  and another point  $P$  on  $l$ . Assuming the theorem of Desargues, we can describe dilatations that fix  $O$ .

**Lemma 6.5.** *Any dilatation fixing  $O$  fixes all lines passing through  $O$ . Further, if  $Q$  is a point on  $l$ , then there is a unique dilatation fixing  $O$  and mapping  $P$  to  $Q$ .*

Thus, dilatations fixing  $O$  can be identified with the affine line  $l$ . The construction of a division ring  $k$  in [1], so that the given affine plane is the standard affine plane over  $k$ , can be summarised as follows.

Consider the set  $k$  of dilatations fixing  $O$ . This becomes a ring as above by regarding dilatations as trace preserving homomorphisms of  $T$ .

**Theorem 6.6.** (See [1].) *The ring  $k$  of dilatations fixing  $O$  is a division ring. The given affine plane is isomorphic to the standard affine plane over  $k$ .*

Thus, as a set  $k$  is an affine line, which can be identified with  $\mathbb{C}$  in our case. We next see that  $k$  is a topological division ring homeomorphic to  $k$ .

## 7. The topological division ring

We first consider the group  $T$  of translations, for which we use the additive notation (for composition). Fix an origin  $O$ . Then  $T$  is in bijective correspondence with the set of points in the affine plane, with a point  $P$  corresponding to the unique translation  $\tau_P$  mapping  $O$  to  $P$ . Observe that  $\tau_{-P} = \tau_P^{-1}$ .

We see that translations form a topological group under this identification.

**Lemma 7.1.** *The group of translations is a topological group with respect to the topology induced from the affine plane by the above identification.*

**Proof.** First consider a pair of points  $P$  and  $Q$  such that  $P \neq O \neq Q$  and  $OP \neq OQ$ . We shall see that  $P + Q$  is given by the parallelogram law.

Note that  $P + Q = \tau_P \circ \tau_Q(O) = \tau_P(Q)$ . As  $\tau_P$  is a dilatation, it follows that  $P + Q$  lies on the line through  $P = \tau_P(O)$  parallel to  $OQ$ . Further, by Lemma 6.1,  $\tau_P(Q)$  lies on the line through  $Q$  parallel to  $OP$ . Hence  $P + Q$  is the intersection of these two lines.

We see similarly that  $P - Q = P + (-Q)$  is given by a parallelogram law, namely it is the intersection of the line through  $P$  parallel to  $OQ$  and the line through  $O$  parallel to  $PQ$ . By the lemmas of Section 4, it follows that both  $P + Q$  and  $P - Q$  vary continuously with  $P$  and  $Q$  provided  $OP \neq OQ$ . For  $Q$  contained in the line  $OP$ , we pick  $R$  not on  $OP = OQ$  and use  $P + Q = (P + R) + (Q - R)$ , which expresses addition as a composition of operations, each of which is continuous by the above.

Similarly, for a point  $P$ , we pick  $R$  not on  $OP$  and use  $-P = R - (R + P)$  to observe that the additive inverse is continuous.  $\square$

We now consider the division ring  $k$ . Pick a point  $1$ . The division ring can be identified with the line  $l$  through  $O$  and  $1$  by identifying a point  $P$  on this line with the dilatation  $\sigma_P$  fixing  $O$  and mapping  $1$  to  $P$ . We shall show that  $k$  is a topological division ring with respect to the topology of  $l$ , using this identification.

Let  $\tau_1$  be the translation mapping  $O$  to  $1$ . The dilatation  $\sigma_P$  acts on  $T$  by the trace preserving homomorphism  $\Phi_P : \tau \mapsto \sigma_P \circ \tau \circ \sigma_P^{-1}$ . In particular, observe that if  $\tau_1 \mapsto \tau$ , then  $\tau(O) = \sigma_P(\tau_1(\sigma_P^{-1}(O))) = \sigma_P(1) = P$ . Thus, we have  $\Phi_P(\tau_1) = \tau_P$ , i.e.,

$$P = \Phi_P(\tau_1)(O).$$

**Lemma 7.2.** *The division ring  $k$  is a topological division ring with respect to the topology induced by the identification of a point  $P$  in  $l$  with the dilatation  $\sigma_P$ .*

**Proof.** First, we consider the additive group of  $k$ , with the points identified with the affine line as above. Temporarily denote the addition on  $l$  from this identification by  $P \oplus Q$ . By the above, this is given by:

$$P \oplus Q = (\Phi_P + \Phi_Q)(\tau_1)(O) = \Phi_P(\tau_1)(O) + \Phi_Q(\tau_1)(O) = P + Q.$$

Thus, addition on  $k$  identified with the affine line  $l$  is the restriction of the addition of the group of translations identified with the affine plane. By Lemma 7.1, addition and the additive inverse are continuous operations.

We now turn to multiplication, which we recall corresponds to composition of the trace preserving homomorphisms. Thus, we have:

$$P \cdot Q = \Phi_P \circ \Phi_Q(\tau_1)(O) = \sigma_P(\sigma_Q(1)) = \sigma_P(Q).$$

By Lemma 5.2,  $P \cdot Q = \gamma_P(Q)$  is continuous as a function of both  $P$  and  $Q$ .

Finally, for  $P \neq O$ , the point  $P^{-1}$  corresponds to the dilatation  $\sigma_{P^{-1}}$  mapping  $O$  to itself and  $P$  to  $1$ . Thus,  $P^{-1} = \sigma_{P^{-1}}(1)$  is continuous as a function of  $P$  by Lemma 5.2.  $\square$

## 8. Proof of the main theorem

We now sketch the rest of the proof of Theorem 1.1.

By the above,  $k$  is a division ring homeomorphic to  $\mathbb{C}$ . By a theorem of Pontryagin [3], any connected, locally compact, division ring is one of  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{H}$ . It follows that  $k = \mathbb{C}$ . Thus, there is a homeomorphism  $\varphi$  from  $\mathbb{C}P^2$  to itself which takes  $J$ -holomorphic curves to lines (with respect to the standard structure). We shall show that  $\varphi$  is smooth.

By results of Gromov, the moduli space  $\mathcal{M}$  of  $J$ -holomorphic curves is a smooth manifold diffeomorphic to  $\mathbb{C}P^2$ . Further, the function associating with a pair of distinct points in  $\mathbb{C}P^2$  the line through these points is smooth, as is the function associating with a pair of distinct lines in  $\mathbb{C}P^2$  (i.e., a pair of distinct points in  $\mathcal{M}$ ) their intersection.

Pick an arbitrary  $J$ -holomorphic curve  $C$  representing the generator of  $H^2(\mathbb{C}P^2)$  and let  $C' = \varphi(C)$ . As  $C'$  is a line, without loss of generality, we can assume  $C \setminus C' = \mathbb{C}^2$ . We shall show that the restriction of  $\varphi$  is a diffeomorphism from the

complement  $A$  of  $C$  to the complement of  $C'$ . Note that, as above,  $A$  is an affine plane over  $\mathbb{C}$  with addition and co-ordinates defined as above.

Now, fix an origin  $O$  in  $A$ . Recall, as in Section 4, that if  $P$  and  $Q$  are points in  $A$  so that  $O$ ,  $P$  and  $Q$  are not collinear, then the sum  $P + Q$  and the difference  $P - Q$  are defined in terms of the operations of intersecting  $J$ -holomorphic curves and of associating with pairs of points the curve passing through them. It follows, again as in Section 4, that the field operations and the vector space operations on  $A$  are smooth. As  $\varphi$  is, by construction, a linear isomorphism between  $A$  as a vector space over  $\mathbb{C}$  defined in terms of these operations and  $\mathbb{C}^2$ , it follows that  $\varphi$  is smooth.

As  $C$  is arbitrary and every point is in the complement of some a curve, it follows that  $\varphi : \mathbb{C}P^2 \rightarrow \mathbb{C}P^2$  is a diffeomorphism. Thus, we obtain a diffeomorphism from  $\mathbb{C}P^2$  with the given complex structure  $J$  to  $\mathbb{C}P^2$  with the standard complex structure  $J_{\text{std}} = i$ , so that lines are mapped to lines.

In particular, for  $p \in \mathbb{C}P^2$ , the almost complex structure  $J$  gives a linear map from  $V = T_p\mathbb{C}P^2$  to itself satisfying  $J^2 = -I$ . We regard  $V$  as a complex vector space using the standard complex structure. The following lemma is elementary.

**Lemma 8.1.** *Let  $J : V \rightarrow V$  be an  $R$ -linear map of a 2-dimensional complex vector space so that  $J^2 = -I$ . Assume that, for all  $v \in V$ , the vectors  $v$  and  $Jv$  are contained in a complex line. Then  $J = \pm i$ .*

As complex lines with respect to  $J$  are complex lines with respect to the standard complex structure, the above lemma applies at each point to show that  $J = \pm J_{\text{std}}$ . In the first case, it follows that this identification is a biholomorphic map. In the second case, we compose with the map  $[z_1 : z_2 : z_3] \mapsto [\bar{z}_1 : \bar{z}_2 : \bar{z}_3]$  to obtain a biholomorphic map.

**Remark 8.2.** An alternative approach to the main result would be to observe that Desargues' theorem implies the existence of symmetries. As almost complex structures are integrable up to first order, these symmetries in turn must imply integrability.

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