Numerical analysis

Approximation by Müntz spaces on positive intervals

Approximation par espaces de Müntz sur un intervalle positif

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The so-called Bernstein operators were introduced by S.N. Bernstein in 1912 to give a constructive proof of Weierstrass’ theorem. We show how to extend his result to Müntz spaces on positive intervals.

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1. Introduction

The famous Bernstein operator $B_k$ of degree $k$ on a given non-trivial interval $[a, b]$, associates with any $F \in C^0([a, b])$ the polynomial function:

$$B_k F(x) := \sum_{i=0}^{k} F\left((1 - \frac{i}{k})a + \frac{i}{k}b\right) B_k^i, \quad x \in [a, b],$$

where $(B_k^0, \ldots, B_k^k)$ is the Bernstein basis of degree $k$ on $[a, b]$, i.e., $B_k^i(x) := \begin{pmatrix} k \\ i \end{pmatrix} (\frac{x-a}{b-a})^i (\frac{b-x}{b-a})^{k-i}$. It reproduces any affine function $U$ on $[a, b]$, in the sense that $B_k U = U$. In [5], S.N. Bernstein proved that, for every function $F \in C^0([a, b])$, $\lim_{k \to +\infty} \|F - B_k F\|_\infty = 0$. In Section 3, we show how this result extends to the class of Müntz spaces (i.e., spaces spanned by power functions) on a given positive interval $[a, b]$, see Theorem 3.1. Beforehand, in Section 2, we briefly remind the reader how to define operators of the Bernstein-type in Extended Chebyshev spaces.

2. Extended Chebyshev spaces and Bernstein operators

Throughout this section, $[a, b]$ is a fixed non-trivial real interval. For any $n \geq 0$, a given $(n + 1)$-dimensional space $E \subset C^0([a, b])$ is said to be an Extended Chebyshev space (for short, EC-space) on $[a, b]$ when any non-zero element of $E$ vanishes at most $n$ times on $[a, b]$ counting multiplicities up to $(n + 1)$.

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Let \( \mathbb{E} \) be an \((n+1)\)-dimensional EC-space on \([a, b]\). Then, \( \mathbb{E} \) possesses bases \((B_0, \ldots, B_n)\) such that, for \( i = 0, \ldots, n, B_i \) vanishes exactly \( i \) times at \( a \) and \((n - i)\) times at \( b \) and is positive on \([a, b]\). We say that such a basis is the Bernstein basis relative to \((a, b)\) if it additionally satisfies \( \sum_{i=0}^{n} B_i = 1 \), where \( 1 \) is the constant function \( \mathbb{1}(x) = 1, x \in [a, b] \). Let us recall that \( \mathbb{E} \) possesses a Bernstein basis relative to \((a, b)\) if and only if, firstly, it contains constants, and secondly the \( n \)-dimensional space \( D\mathbb{E} := \{ F : F \mid F \in \mathbb{E} \} \) is an EC-space on \([a, b]\). Note that the second property is not an automatic consequence of the first one, see [8] and other references therein.

As an instance, given any pairwise distinct \( \lambda_0, \ldots, \lambda_k \), the so-called Müntz space \( M(\lambda_0, \ldots, \lambda_k) \), spanned over a given positive interval \([a, b]\) \((i.e., a > 0)\) by the power functions \( x^{\lambda_i}, 0 \leq i \leq k \), is a \((k+1)\)-dimensional EC-space on \([a, b]\). If \( \lambda_0 = 0 \), since \( D(M(\lambda_0, \ldots, \lambda_k)) = M(\lambda_1 - 1, \ldots, \lambda_k - 1) \), the space \( M(\lambda_0, \ldots, \lambda_k) \) possesses a Bernstein basis relative to \((a, b)\).

For the rest of the section, we assume that \( \mathbb{E} \subset C^0([a, b]) \) contains constants and that \( D\mathbb{E} \) is an \((n\)-dimensional) EC-space on \([a, b]\). We denote by \((B_0, \ldots, B_n)\) the Bernstein basis relative to \((a, b)\) in \( \mathbb{E} \).

**Definition 2.1.** A linear operator \( \mathbb{B} : C^0([a, b]) \rightarrow \mathbb{E} \) is said to be a **Bernstein operator based on** \( \mathbb{E} \) when, firstly it is of the form:

\[
\mathbb{B} F := \sum_{i=0}^{k} F(\xi_i)B_i, \quad \text{for some } a = \xi_0 < \xi_1 < \cdots < \xi_n = b, \tag{2}
\]

and secondly it reproduces a two-dimensional EC-space \( U \) on \([a, b]\), in the sense that \( \mathbb{B} V = V \) for all \( V \in U \).

Any Bernstein operator \( \mathbb{B} \) is positive \((i.e., F \geq 0 \implies \mathbb{B} F \geq 0)\) and shape preserving due to the properties of Bernstein bases in EC-spaces, see [8]. Note that everything concerning Bernstein-type operators in EC-spaces with no Bernstein bases can be deduced from Bernstein operators as defined above [8,9].

**Theorem 2.2.** Given \( n \geq 2 \), let \( \mathbb{E} \subset C^0([a, b]) \) contain constants. Assume that \( D\mathbb{E} \) is an \( n \)-dimensional EC-space on \([a, b]\). For a function \( U \in \mathbb{E} \), expanded in the Bernstein basis relative to \((a, b)\) as \( U := \sum_{i=0}^{n} u_i B_i \), the following properties are equivalent:

(i) \( u_0, \ldots, u_n \) form a strictly monotone sequence;

(ii) there exists a nested sequence \( E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n := \mathbb{E} \), where \( E_1 := \text{span}(1, u) \) and where, for \( i = 1, \ldots, n-1 \), \( E_i \) is an \((i+1)\)-dimensional EC-space on \([a, b]\);

(iii) there exists a Bernstein operator based on \( \mathbb{E} \) which reproduces \( U \).

In [8] it was proved that there exists a one-to-one correspondence between the set of all Bernstein operators based on \( \mathbb{E} \) and the set of all two-dimensional EC-spaces \( U \) they reproduce. In particular, if (i) holds, then the unique Bernstein operator based on \( \mathbb{E} \) reproducing \( U \) is defined by (2) with:

\[
\xi_i := U^{-1}(u_i), \quad 0 \leq i \leq n. \tag{3}
\]

Note that this is meaningful because (i) implies the strict monotonicity of \( U \) on \([a, b]\). Condition (ii) of Theorem 2.2 yields the following corollary.

**Corollary 2.3.** Given an integer \( n \geq 1 \), consider a nested sequence:

\[
E_n \subset E_{n+1} \subset \cdots \subset E_p \subset E_{p+1} \subset \cdots, \tag{4}
\]

where \( E_n \) contains constants and for any \( p \geq n \), \( D E_p \) is a \( p \)-dimensional EC-space on \([a, b]\). Let \( U \in E_n \) be a non-constant function reproduced by a Bernstein operator \( \mathbb{E}_n \) based on \( \mathbb{E}_n \). Then, \( U \) is also reproduced by a Bernstein operator \( \mathbb{E}_p \) based on \( \mathbb{E}_p \) for any \( p > n \).

**Remark 2.4.** In the situation described in Corollary 2.3, a natural question arises: given \( F \in C^0([a, b]) \), does the sequence \( B_k F, k \geq n \), converges to \( F \) in \( C^0([a, b]) \) equipped with the infinite norm? Obviously, for this to be true for any \( F \in C^0([a, b]) \), it is necessary that \( \bigcup_{k \geq n} E_k \) be dense in \( C^0([a, b]) \). The example of Müntz spaces proves that this is not always satisfied.

3. Müntz spaces over positive intervals

Throughout this section, we consider a fixed positive interval \([a, b]\), a fixed infinite sequence of real numbers \( \lambda_k, k \geq 0 \), assumed to satisfy:

\[
0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \quad \lim_{k \to +\infty} \lambda_k = +\infty. \tag{5}
\]

We are concerned with the corresponding nested sequence of Müntz spaces:

\[
M(\lambda_0) \subset M(\lambda_0, \lambda_1) \subset \cdots \subset M(\lambda_0, \ldots, \lambda_k) \subset M(\lambda_0, \ldots, \lambda_k, \lambda_{k+1}) \subset \cdots \tag{6}
\]
Given any $n \geq 1$, for each $k \geq n$, we can select a Bernstein operator $B_k$ based on $M(\lambda_0, \ldots, \lambda_k)$. Assume the sequence $B_k$, $k \geq n$, to satisfy:

$$
\lim_{k \to +\infty} \|F - B_k F\|_\infty = 0 \quad \text{for any } F \in C^0([a, b]).
$$

(7)

Then, the union of all spaces $M(\lambda_0, \ldots, \lambda_k)$, $k \geq 0$, is dense in $C^0([a, b])$ equipped with the infinite norm. As is well known, this holds if and only if the sequence (5) fulfills the so-called Müntz density condition below [4,6]:

$$
\sum_{i \geq 1} \frac{1}{\lambda_i} = +\infty.
$$

(8)

As an instance, the Müntz condition (8) is satisfied when $\lambda_k = \ell + 1$ for all $k \geq 1$. This case was addressed in [8]. Convergence – in the sense of (7) – was proved there under the assumption that each $B_k$ reproduced the function $x^\ell$. This convergence result includes the classical Bernstein operators [5] obtained with $\ell = 0$. Below we extend it to the general interesting situation of sequences of Müntz Bernstein operators $B_k$ all reproducing the same two-dimensional EC-space (see Remark 2.4).

**Theorem 3.1.** Given $n \geq 1$, let $E_1 \subset M(\lambda_0, \ldots, \lambda_n)$ be a two-dimensional EC-space reproduced by a Bernstein operator $B_k$ based on $M(\lambda_0, \ldots, \lambda_k)$ for any $k \geq n$. Then, if the Müntz density condition (8) holds, the sequence $B_k$, $k \geq n$, converges in the sense of (7).

Before starting the proof, let us introduce some notations. For $k \geq 1$, denote by $(B_{k,0}, \ldots, B_{k,k})$ the Bernstein basis relative to $(a, b)$ in the Müntz space $M(\lambda_0, \ldots, \lambda_k)$. We consider the functions:

$$
U^*(x) = x^{\lambda_1}, \quad V_p(x) := x^{\lambda_p}, \quad p \geq 2, \ x \in [a, b],
$$

expanded in the successive Bernstein bases as:

$$
U^* = \sum_{i=0}^{k} u^*_{k,i} B_{k,i} \quad \text{for all } k \geq 1, \quad V_p = \sum_{i=0}^{k} v_{p,k,i} B_{k,i} \quad \text{for all } k \geq p.
$$

(9)

With these notations, the key-point to prove Theorem 3.1 is the following lemma, for the proof of which we refer to [1], see also [2].

**Lemma 3.2.** Assume that the Müntz density condition (8) holds. Then, we have:

$$
\lim_{k \to +\infty} \max_{0 \leq i \leq k} \left|(u^*_{k,i})^{\frac{1}{1+i}} - v_{p,k,i}\right| = 0 \quad \text{for all } p \geq 2.
$$

(10)

**Proof of Theorem 3.1.** Let us start with the simplest example $n = 1$. Then, $E_1 = \text{span}(1, U^*)$. For each $k \geq 1$, the unique operator $B^*_{k}$ which reproduces $E_1$ is given by:

$$
B^*_{k} F := \sum_{i=0}^{k} F(\zeta^*_{k,i}) B_{k,i}. \quad \text{with, for } i = 0, \ldots, k, \ z^*_{k,i} := (u^*_{k,i})^{\frac{1}{1+i}}.
$$

(11)

According to Korovkin’s theorem for positive linear operators [7], we just have to select a function $F$ so that $1, U^*, F$ span a three-dimensional EC-space on $[a, b]$ and prove that $\lim_{k \to +\infty} \|F - B^*_{k} F\|_\infty = 0$ for this specific $F$. We can thus choose for instance $F := V_2$. Actually, we will more generally prove the result with $F = V_p$, for any $p \geq 2$. Using (9) and (11), we obtain, for any $k \geq p$,

$$
\|B^*_k V_p - V_p\|_\infty = \left\|\sum_{i=0}^{k} (V_p(\zeta^*_{k,i}) - v_{p,k,i}) B_{k,i}\right\|_\infty \leq \max_{0 \leq i \leq k} |V_p(\zeta^*_{k,i}) - v_{p,k,i}|.
$$

(12)

On account of (11), Lemma 3.2 yields the expected result:

$$
\lim_{k \to +\infty} \|B^*_k V_p - V_p\|_\infty = 0 \quad \text{for each } p \geq 2.
$$

- We now assume that $n > 1$. Select a strictly increasing function $U \in E_1$. Condition (ii) of Theorem 2.2 enables us to select a function $V \in M(\lambda_0, \ldots, \lambda_n)$ so that the functions $1, U, V$ span a three-dimensional EC-space on $[a, b]$. For any $k \geq n$, expand $U, V$ as:
\[ U = \sum_{i=0}^{k} u_{k,i} B_{k,i}, \quad V = \sum_{i=0}^{k} v_{k,i} B_{k,i}. \]

We know that, for each \( k \geq n \), the sequence \((u_{k,0}, \ldots, u_{k,k})\) is strictly increasing, and that the Bernstein operator \( B_k \) is defined by formula (2) with \( \zeta_{k,i} := U^{-1}(u_{k,i}) \) for \( i = 0, \ldots, k \). Via expansions of \( U \) and \( V \) in the basis \((1, U^*, V_2, \ldots, V_n)\) of the Müntz space \( M(\lambda_0, \ldots, \lambda_n) \), Lemma 3.2 readily proves that:

\[ \lim_{k \to +\infty} \max_{0 \leq i \leq k} \left| U(\zeta_{k,i}^*) - u_{k,i} \right| = 0 = \lim_{k \to +\infty} \max_{0 \leq i \leq k} \left| V(\zeta_{k,i}^*) - v_{k,i} \right|. \] \hspace{1cm} (13)

The left part in (13) can be written as \( \lim_{k \to +\infty} \max_{0 \leq i \leq k} \left| U(\zeta_{k,i}^*) - U(\zeta_{k,i}) \right| = 0 \). On this account, the uniform continuity of the function \( V \circ U^{-1} \) and the right part in (13) prove that \( \lim_{k \to +\infty} \max_{0 \leq i \leq k} \left| V(\zeta_{k,i}) - v_{k,i} \right| = 0 \), thus implying that \( \lim_{k \to +\infty} \left\| B_k V - V \right\|_{\infty} = 0 \). By Korovkin’s theorem, (7) is satisfied.

**Remark 3.3.** Given \( n \geq 2 \), one can apply Theorem 3.1 with \( \mathcal{E}_1 := \text{span}(1, V_n) = M(\lambda_0, \lambda_n) \), due to the nested sequence of Müntz spaces \( M(\lambda_0, \lambda_1, \ldots, \lambda_{i-1}, \lambda_n) \) for \( 1 \leq i \leq n \). Note that Theorem 3.1 contains in particular the Bernstein-type result expected in [3].

**References**


