Partial differential equations

Cauchy problem for effectively hyperbolic operators with triple characteristics

Problème de Cauchy pour des opérateurs effectivement hyperboliques ayant des caractéristiques triples

Enrico Bernardi\textsuperscript{a}, Antonio Bove\textsuperscript{b}, Vesselin Petkov\textsuperscript{c}

\textsuperscript{a} Dipartimento di Scienze Statistiche, Università di Bologna, Viale Filopanti 5, 40126 Bologna, Italy
\textsuperscript{b} Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40127 Bologna, Italy
\textsuperscript{c} Université Bordeaux I, Institut de Mathématiques, 351, cours de la Libération, 33405 Talence, France

E-mail addresses: bernardi@unibo.it (E. Bernardi), antonio.bove@unibo.it (A. Bove), petkov@math.u-bordeaux1.fr (V. Petkov).

© 2013 Published by Elsevier Masson SAS on behalf of Académie des sciences.

Article history:
Received 22 August 2013
Accepted 11 October 2013
Available online 30 December 2013
Presented by Jean-Michel Bony

We study a class of third-order effectively hyperbolic operators \(P\) in \(G = \{(t, x): 0 \leq t \leq T, x \in U \subseteq \mathbb{R}^n\}\) with triple characteristics at \(\rho = (0, x_0, \xi), \xi \in \mathbb{R}^n \setminus \{0\}\). V. Ivrii introduced the conjecture that every effectively hyperbolic operator is \textit{strongly hyperbolic}, that is the Cauchy problem for \(P + Q\) is locally well posed for any lower-order terms \(Q\). For operators with triple characteristics, this conjecture was established [3] in the case when the principal symbol of \(P\) admits a factorization as a product of two symbols of principal type. A strongly hyperbolic operator in \(G\) could have triple characteristics in \(G\) only for \(t = 0\) or for \(t = T\). The operators that we investigate have a principal symbol which in general is not factorizable and we prove that these operators are strongly hyperbolic if \(T\) is small enough.

© 2013 Published by Elsevier Masson SAS on behalf of Académie des sciences.

On étudie une classe d’opérateurs effectivement hyperboliques \(P\) dans \(G = \{(t, x): 0 \leq t \leq T, x \in U \subseteq \mathbb{R}^n\}\) ayant des caractéristiques triples pour \(\rho = (0, x_0, \xi), \xi \in \mathbb{R}^n \setminus \{0\}\). V. Ivrii a introduit la conjecture selon laquelle chaque opérateur effectivement hyperbolique est fortement hyperbolique, c’est-à-dire telle que le problème de Cauchy pour \(P + Q\) soit localement bien posé pour tout opérateur \(Q\) d’ordre inférieur à celui de \(P\). Pour des opérateurs ayant des caractéristiques triples, cette conjecture a été démontrée [3] pour le cas où le symbole principal de \(P\) admet une factorisation comme produit de deux symboles du type principal. Un opérateur fortement hyperbolique pourrait avoir des caractéristiques triples seulement pour \(t = 0\) ou pour \(t = T\). Les opérateurs que nous examinons ont en général un symbole principal qui n’est pas factorisable, et nous prouvons qu’ils sont fortement hyperboliques si \(T\) est suffisamment petit.

© 2013 Published by Elsevier Masson SAS on behalf of Académie des sciences.
1. Introduction and main result

Consider a differential operator \( P(t, x, D_t, D_x) = \sum_{|\alpha|+|\beta| \leq m} c_{\alpha, \beta}(t, x) D_t^\alpha D_x^\beta \). \( D_t = -i\partial_t \), \( D_x = -i\partial_x \) of order \( m \) with \( C^\infty \) coefficients \( c_{\alpha, \beta}(t, x), t \in \mathbb{R}, x \in \mathbb{R}^n \). Denote by \( p_m(t, x, \tau, \xi) \) the principal symbol of \( P \). We assume that \( c_{\alpha, \beta}(t, x) = 1 \) for all \( (t, x) \). Let \( \Omega \subset \mathbb{R}^{n+1}_+ \) be an open set and let \( \Omega^-_\eta = \Omega \cap \{ t \leq \eta \}, \Omega^+_\eta = \Omega \cap \{ t \geq \eta \}, \eta = \Omega \cap \{ 0 \leq t \leq T \} \). Set \( p_m(t, x, D_t, D_x) = p_m(t, x, D_t, D_x) \).

**Definition 1.** We say that the Cauchy problem:

\[
P u = f \quad \text{in} \quad \Omega \cap \{ t < T \}, \quad \text{supp} \, u \subset \overline{G}
\]

is well posed in \( G \) if: (i) (existence) for every \( f \in C_0^\infty(\Omega) \), \( \text{supp} \, f \subset \overline{\Omega}_0^- \) there exists a solution \( u \in D'(\Omega) \) satisfying (1.1),

(ii) (uniqueness) if \( u \in D'(\Omega) \) satisfies (1.1), then for every \( s, 0 \leq s \leq T \), if \( Pu = 0 \) in \( \Omega^-_s \), then \( u = 0 \) in \( \Omega^-_s \).

A necessary condition for the well-posedness of the Cauchy problem (WPC) is the hyperbolicity of the operator \( P \) in \( G \) (see [4] and the references cited there). This means that for every \( (t_0, x_0, \xi) \in G \times \mathbb{R}^n \setminus \{0\} \), the equation \( p_m(t_0, x_0, \tau, \xi) = 0 \) with respect to \( \tau \) has only real roots \( \tau = \lambda_j(t_0, x_0, \xi) \).

**Definition 2.** We say that the operator \( P \) with principal symbol \( p_m \) is strongly hyperbolic in \( G \) if for every point \( z_0 = (t_0, x_0) \in G \) there exists a neighborhood \( U \) of \( z_0 \) \( T(U) > 0 \) and \( T_0 > 0 \) \( (T_0 < T \text{ if } t_0 = T \text{ and } T_0 = 0 \text{ if } t_0 = 0) \) such that the Cauchy problem (1.1) for the operator \( L = P_m(t, x, D_t, D_x) + Q_{m-1}(t, x, D_t, D_x) \) is well posed in \( U \cap \{ s \leq t \leq T(U) \} \) for every \( T_0 \leq s < T(U) \) and for any operator \( Q_{m-1}(t, x, D_t, D_x) \) of order less or equal to \( m - 1 \).

In the following, we switch to a different notation for the sake of simplicity and denote \( t = x_0, x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \). The dual variables are denoted by \( \xi = (\xi_0, \xi_1, \ldots, \xi_n) = (\xi_0, \xi) \). Given a symbol \( p(x, \xi) \), let \( \Sigma(p) = \{ z \in T^*G \setminus \{0\} \} \). In the case \( \Sigma(p) = \emptyset \), the operator is of principal type and \( P \) is strongly hyperbolic in \( G \) (see Section 23.4 in [2] and [3]). Passing to the case \( \Sigma_1(p_m) \neq \emptyset \), for \( (\hat{x}, \hat{\xi}) \in \Sigma_1(p_m) \), we are led to define the fundamental matrix:

\[
F_p(\hat{x}, \hat{\xi}) = \begin{pmatrix}
  p_{\xi, \xi}(\hat{x}, \hat{\xi}) & p_{\xi, x}(\hat{x}, \hat{\xi}) & p_{x, x}(\hat{x}, \hat{\xi}) \\
  -p_{\xi, \xi}(\hat{x}, \hat{\xi}) & -p_{\xi, x}(\hat{x}, \hat{\xi}) & -p_{x, x}(\hat{x}, \hat{\xi})
\end{pmatrix}
\]

If \( P \) is hyperbolic in \( G \) and \( (\hat{x}, \hat{\xi}) \) is a critical point of \( p_m(x, \xi) \), then \( F_p(\hat{x}, \hat{\xi}) \) has at most two non-vanishing real simple eigenvalues \( \mu \) and \( -\mu \) and all other eigenvalues \( \mu_j \) are purely imaginary, that is \( \text{Re} \, \mu_j = 0 \). Moreover, under canonical transformations, \( F_p \) is transformed into a similar matrix and the eigenvalues of \( F_p \) remain invariant. The existence of non-vanishing real eigenvalues of \( F_p \) is a necessary condition for strong hyperbolicity. More precisely, let \( p_m(x, \xi) = \sum_{|\alpha| = m-1} c_{\alpha}(x) \xi^\alpha \) and let \( p_{m-1}(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=0}^n \partial^2 p_m \partial_{\xi j}^2(x, \xi) \) be the subprincipal symbol of \( P \), which is invariantly defined for \( (x, \xi) \in \Sigma_1(p_m) \). It was proved in [4] that if \( F_p \) has no non-vanishing real eigenvalues, then \( \text{Im} \, p_{m-1}(\hat{x}, \hat{\xi}) = 0 \) is a necessary condition for (WPC). A hyperbolic operator with principal symbol \( p_m(\xi, \xi) \) will be called **effectively hyperbolic** if at every point \( (\hat{x}, \hat{\xi}) \in \Sigma_1(p_m) \), the fundamental matrix \( F_p(\hat{x}, \hat{\xi}) \) has two non-vanishing real eigenvalues. V. Ivrii introduced the following.

**Conjecture.** A hyperbolic operator is strongly hyperbolic if and only if it is effectively hyperbolic.

For operators with at most double characteristics the sufficient part of the above conjecture has been established in a number of papers: see [5,7] for two different proofs as well as [3,6,8] for some special classes of operators. According to Theorem 3 and Corollary 3 in [4], a strongly hyperbolic operator may have triple characteristics only for \( t = 0 \) or \( t = T \) and in this Note we deal with the case when this happens for \( t = 0 \). More precisely, we examine operators having the form:

\[
P = D_t^3 + \sum_{j=1}^3 q_j(t, x, D_x) D_t^{3-j} + \sum_{j=1}^3 r_j(t, x, D_x) D_t^{2-j} + \sum_{j=1}^3 r_j(t, x) D_t^{1-j} + m_1(t, x, D_x) + c_0(t, x).
\]

Here \( q_j(t, x, D_x), j = 1, 2, 3 \), are differential operators with \( C^\infty \) coefficients and real-valued symbols \( q_j(t, x, \xi) \) that are homogeneous polynomials of order \( j \) in \( \xi, r_j(t, x, D_x), j = 1, 2 \), are differential operators with \( C^\infty \) coefficients and symbols \( r_j(t, x, \xi) \) homogeneous of order \( j \) with respect to \( \xi \); \( m_1(t, x, D_x) \) is a first order differential operator with \( C^\infty \) coefficients and \( r_j(t, x, \xi) \) are \( C^\infty \) functions. Consider the symbols \( \Delta_1 = 27q_3 - 9q_1q_2 + 2q_1^2, \Delta_0 = q_1^2 - 3q_2, \Delta = -\frac{1}{27}(\Delta_1^2 - 4\Delta_0^3) \). The symbol \( \Delta \) is the discriminant of the equation \( p_3 = 0 \) with respect to \( \tau \) and we have three real roots for \( \tau \geq 0 \) if and only if \( \Delta \geq 0 \). The symbol \( \Delta_0 \) is the discriminant of the equation \( \partial_\rho p_3 = 0 \) and if we have a triple root at \( \rho \approx (0, x_0, \xi) \), we get \( \Delta_0(\rho) = \Delta_1(\rho) = 0 \). Moreover, the hyperbolicity of \( p_3 \) for \( \tau \geq 0 \) implies \( d\xi, \Delta(\rho) = 0 \).
We make the following assumptions:

\( (H_0) \) The roots of the equation \( p_3(t, x, \tau, \xi) = 0 \) with respect to \( \tau \) are real for all \((t, x) \in \mathbb{G}, \xi \in \mathbb{R}^n \setminus \{0\} \).

\( (H_1) \) If the equation \( p_3(0, x, \tau, \xi) = 0 \) with respect to \( \tau \) has a triple root \( \tau = \lambda(0, x_0, \xi_0) \) for \( t = 0, x_0 \in U, \xi_0 \in \mathbb{R}^n \setminus \{0\} \), then we have a triple root \( \tau = \lambda(0, x_0, \xi) \) for \( (0, x_0, \xi), \forall \xi \in \mathbb{R}^n \setminus \{0\} \) and the fundamental matrix \( F_{p_3}(0, x_0, \lambda(0, x_0, \xi), \xi) \) of \( p_3 \) has non-zero real eigenvalues \( \pm \mu(x_0, \xi) \) for \( \xi \in \mathbb{R}^n \setminus \{0\} \).

\( (H_2) \) If \( (H_1) \) holds for \((0, x_0, \xi) \), then there exists an open neighborhood \( U_{x_0} \) of \( x_0 \) such that \( \Delta_{1}(0, x, \xi) = (\partial_{t} \Delta_{1})(0, x, \xi) = 0 \) for \( x \in U_{x_0} \) and \( \xi \in \mathbb{R}^n \setminus \{0\} \).

Our main result is the following.

**Theorem 3.** Assume the hypothesis \( (H_0)-(H_2) \) satisfied. Then there exists a open neighborhood \( V_{x_0} \subset U_{x_0} \) such that for \( T > 0 \) sufficiently small, the operator \( P \) is strongly hyperbolic in \((t, x): 0 \leq t \leq T, x \in V_{x_0}\).

For operators with triple characteristics, the conjecture of Ivrii has been proved in [3] in the case when the principal symbol \( p_3 \) of \( P \) admits a factorization \( p_3 = ((\tau - \beta(t, x, \xi))^2 - D(t, x, \xi))^{1/2} \) with smooth real valued symbols \( \beta, \gamma \) homogeneous of order 1 in \( \xi \) and real-valued symbol \( D \geq 0, t > 0 \) homogeneous of order 2 in \( \xi \). On the other hand, it was proved in [1] that for symbols having the form (2.2) given below with \( b_3(0, x_0, \xi_0) \neq 0 \), a factorization in a neighborhood of \((0, x_0) \) is possible if and only if \( \alpha(x, \xi_0) = 0 \) for \( x \) in a neighborhood of \( x_0 \).

2. Idea of the proof of Theorem 3

To prove **Theorem 3**, we obtain an a priori estimate with a loss of regularity of order \( 2N/3 - 2 \) for the operator \( P \). We choose \( N = \frac{12}{7} \Pi + N_0 \), with \( N_0 \) an integer and:

\[
\Pi = \frac{2}{3} \sup_{x \in U_{x_0}, \|\xi\| = 1} |p'_2(0, x, \lambda(x, \xi), \xi)(\mu(x, \xi))^{-1}|, \tag{2.1}
\]

where \( p'_2(t, x, \tau, \xi) \) is the subprincipal symbol of \( p_3 \), \( \lambda(0, x, \xi) \) denotes the triple root of \( p_3 = 0 \) and \( \mu(x, \xi) \) is the non-vanishing eigenvalue of \( F_{p_3} \). Moreover, the integer \( N_0 \) depends only on \( p_3(0, x, \lambda(0, x, \xi), \xi) \), but we are not going to clarify the optimal value of \( N_0 \).

**Step 1.** By a change of variables \((x, t) \), we reduce the analysis to the case when the principal symbol has the form:

\[
p_3(t, x, \tau, \xi) = \tau^3 - (a_2(t, x, \xi) + \alpha(x, \xi)) \tau + t^2 b_3(t, x, \xi). \tag{2.2}
\]

where \( a_2(t, x, \xi), \alpha(x, \xi), \) are real valued symbols homogeneous with respect to \( \xi \) of order 2 and \( a_2(t, x, \xi) \geq \delta |\xi|^2, \delta > 0, \alpha(x, \xi) \geq 0 \), while \( b_3(t, x, \xi) \) is homogeneous of order 3 in \( \xi \). Next, we introduce the scaling \( t = \epsilon^{2/3} t, x = \epsilon y, \epsilon > 0 \), and we transform \( P \) into an operator \( \mathcal{P} \) with respect to \((y, s) \). Since we are interested in showing that the Cauchy problem is well posed for sufficiently small \( t \) and since \( P \) is strictly hyperbolic for \( t \) positive and small enough, we can investigate the operator \( \mathcal{P} \). We use again the notation \((t, x) \) for the new variables. The symbols \( a_2, \alpha, b_3 \) are transformed to symbols \( a_2' = a_2(\epsilon^{2/3} t, \epsilon x, \xi), \alpha' = \alpha(\epsilon x, \xi), b_3' = b_3(\epsilon^{2/3} t, \epsilon x, \xi) \) and this is important for the pseudodifferential calculus which we develop. Eventually we choose \( \epsilon = o(N^{-1}) \), where \( N \) is given above, so that \( 0 < \epsilon_0 \leq \epsilon N \ll 1 \).

**Step 2.** We choose a time function \( f = \frac{t}{3} + \frac{|\xi|}{\epsilon^{2/3}} \), \( |\xi| = (1 + |\xi|^2)^{1/2} \) which plays an important role in the calculus of pseudodifferential operators with order function \( \mathcal{m}_N = f^{-N}(t, \xi) \) and metric \( g_{(x, \xi)} = \epsilon^4 |dx|^2 + (|\xi|^{-2} |d\xi|^2) \). The issue of this choice is that the commutator \([a_2'(t, x, D_x), f^{-N}(t, D_x)]\) has symbol in the space \( S(f^{-N} \epsilon N(t, \xi), g^*_{f}) \), so \([a_2'(t, x, D_x), f^{-N}(t, D_x)]f^{N}(t, D_x) \) becomes a first-order operator whose norm is not depending on \( N \) and, in particular, on \( \Pi \).

**Step 3.** In order to derive energy estimates, we multiply \( \mathcal{P} u \) by the multiplier operator:

\[
Mu = \psi(t) \left(D_2^2 u - \frac{1}{3} \{t a'_2(t, x, D_x) u + \epsilon^{-2/3} \alpha'(x, D_x) u \} \right),
\]

where \( \psi(t) = \frac{\epsilon^{2/3} t}{t}, \lambda > 0 \), and then study the expression \(-2 \text{Im}(f^{-2N}(t, D_x) \mathcal{P} u, Mu)\), \((, )\) denoting the scalar product in \( L^2(\mathbb{R}^n) \) and \( u \in C_c^\infty([0, T] \times U) \). The above expression modulo lower-order terms is a sum of fifteen terms and we make a quite detailed analysis of all these terms. The purpose is to find, by integration by parts, “positive” terms with a big coefficient of order \( O(N) \) which will absorb in the energy estimate the contributions with “indefinite” (possibly negative) sign.

**Step 4.** The positive terms in the energy estimates come from the expression \( \partial_t \delta_N(u) + \frac{2N}{3} \delta_{N+1/2}(u) \), where:
To overcome this difficulty, we use the key inequality
\[ \| f^{-N}u'' \|_0^2 + \frac{2}{3} \text{Re}(f^{-2N}(ta'_x + e^{-2/3}\alpha')u', u') + \frac{1}{3} \| f^{-N}(ta''_x + e^{-2/3}\alpha')u \|_0^2 \]
\[ + \frac{2}{3} \text{Re}(f^{-2N}u'\cdot (ta'_x + e^{-2/3}\alpha')u) + \epsilon^{1/3}t e^{-2\lambda t}2 \text{Im}(f^{-2N}b_2 u, u) \].  
\[(2.3)\]

The quantity in the second line above, involving powers of \( \epsilon \) and \( t \), is a lower-order perturbation.

**Step 5.** To control the terms involving \( \| f^{-N}u(t, \cdot) \|_2 \) norms and \( k = N, N + 1/2 \), we would like to exploit the terms with \( H^2(\mathbb{R}^n) \) norms having large coefficients. However, the latter terms have factors \( t \) so that \( tN \) becomes close to 0. To overcome this difficulty, we use the key inequality \( \langle \xi \rangle^A f^{-k} \leq t \langle \xi \rangle^B f^{-k-1} + \langle \xi \rangle^C f^{-k-3} \). Thus, for example, we have an estimate \( \| f^{-N}u \|_2^2 \leq t \| f^{N-1/2}u \|_2^2 + \| f^{-N-3/2}u \|_2^2 \). For \( \| f^{-N-1/2}u \|_2^2 \), we must apply the key inequality twice to gain \( t^2 \) factors and this leads to norms of the form \( \| f^{-N-k/2}u \|_0^2 \). Following this method, we also obtain negative terms with coefficients \( t \) and \( t^2 \), as well as a number of other negative terms involving norms with weights \( f^{-N-k/2} \) and \( f^{-N-k/3} \); these have to be absorbed too. To be able to handle such terms, we use some differential inequalities involving \( \psi \) and \( -\psi' \) and we take the parameter \( \lambda \) sufficiently large. Finally, we obtain:

**Theorem 4.** Assume that \( (D^2u)(t, x) = (Dt)(t, x) = u(t, x) = 0 \) and \( 0 \leq t \leq s \leq T \) with a small \( T > 0 \). Then for every \( p \in \mathbb{R} \) there exist \( \lambda_p \) and a constant \( C_p \) so that for \( \lambda \geq \lambda_p, N = \frac{12}{7} \pi + N_0 \) and \( u \in C_\infty^0(\mathbb{R}^{n+1}) \), we have the estimate:

\[ \lambda \int t e^{-2\lambda s - 2N \log(1+s)} \left( \sum_{k=0}^2 \| \delta^k u(s, \cdot) \|_{2^{-k+p}}^2 + \right. \left. \sum_{k=0}^2 \| \delta^k u(s, \cdot) \|_{(2^{-k+p})}^2 \right) ds \]
\[ \leq C_p \int t e^{-2\lambda s - 2N \log(1+s)} \left( \right. \left. \| u(s, \cdot) \|_{2^{n/3+p}}^2 \right) ds. \]
\[(2.4)\]

where \( \| \cdot \|_{m} \) is the \( H^m \) norm in \( \mathbb{R}^n \).

The details of the proofs are included in [1].

**References**


