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Weighted-composition operators on $N_p$-spaces in the ball

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ABSTRACT

In this Note, we introduce $N_p$-spaces, some kind of Bergman-type spaces, of holomorphic functions in the unit ball of $\mathbb{C}^n$. Basic properties of these spaces are provided. We study weighted-composition operators between $N_p$-spaces and the spaces $A^{-q}$ and obtain, in particular, criteria for boundedness and compactness of such operators.

RéSUMÉ

Dans cette Note, nous introduisons les espaces $N_p$, qui sont des analogues d'espaces de Bergman de fonctions holomorphes sur la boule unité de $\mathbb{C}^n$. Les propriétés de base de ces espaces sont données. Nous étudions ensuite les opérateurs de composition à poids de $N_p$ dans $A^{-q}$, et nous obtenons, en particulier, des critères pour que ces opérateurs soient bornés ou compacts.

1. Introduction

1.1. Notation and definitions

Let $\mathbb{B}$ be the unit ball for the Euclidean norm in the complex vector space $\mathbb{C}^n$; $\mathcal{O}(\mathbb{B})$ denotes the space of functions that are holomorphic in $\mathbb{B}$, with the compact-open topology, and $H^\infty(\mathbb{B})$ denotes the Banach space of bounded holomorphic functions on $\mathbb{B}$ with the norm $\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|$.

If $z = (z_1, z_2, \ldots, z_n)$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}^n$, then $(z, \zeta) = z_1 \zeta_1 + \cdots + z_n \zeta_n$ and $|z| = (z_1 \bar{\zeta}_1 + \cdots + z_n \bar{\zeta}_n)^{1/2}$.

If $X$ and $Y$ are two topological vector spaces, then the symbol $X \hookrightarrow Y$ means the continuous embedding of $X$ into $Y$.

Let $p > 0$, the Beurling-type space (sometimes also called the Bergman-type space) $A^{-p}(\mathbb{B})$ in the unit ball is defined as:

$$A^{-p}(\mathbb{B}) := \{f \in \mathcal{O}(\mathbb{B}) : |f|_p = \sup_{z \in \mathbb{B}} |f(z)| (1 - |z|^2)^{p/2} < \infty\}.$$ 

For $\varphi$ a holomorphic self mapping of $\mathbb{B}$ and a holomorphic function $u : \mathbb{B} \to \mathbb{C}$, the linear operator $W_{u,\varphi} : \mathcal{O}(\mathbb{B}) \to \mathcal{O}(\mathbb{B})$:

$$W_{u,\varphi}(f)(z) = u(z) \cdot (f \circ \varphi(z)), \quad f \in \mathcal{O}(\mathbb{B}), \ z \in \mathbb{B},$$

is called the weighted-composition operator with symbols $u$ and $\varphi$. Observe that $W_{u,\varphi}(f) = M_u C_\varphi(f)$, where $M_u(f) = uf$, is the multiplication operator with symbol $u$, and $C_\varphi(f) = f \circ \varphi$ is the composition operator with symbol $\varphi$. If $u$ is identically 1, then $W_{u,\varphi} = C_\varphi$, and if $\varphi$ is the identity, then $W_{u,\varphi} = M_u$.

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Composition operators and weighted composition operators acting on spaces of holomorphic functions in the unit disk \( \mathbb{D} \) of the complex plane have been studied quite well. We refer the readers to the monographs [1,6] for detailed information. Composition operators on \( A^{-p}(\mathbb{D}) \) have also been intensively studied (see, e.g., [2] and references therein).

1.2. \( \mathcal{N}_p \)-spaces in the unit ball

Given a point \( a \in \mathbb{B} \), we can associate with it the following automorphism \( \Phi_a \in \text{Aut}(\mathbb{B}) \) (see, e.g., [5, pp. 25–27]):

\[
\Phi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle},
\]

where \( P_a \) is the orthogonal projection of \( \mathbb{C}^n \) on a subspace \([a]\) generated by \( a \), that is,

\[
P_a z = \begin{cases} 
0, & \text{if } a = 0, \\
\frac{\langle z, a \rangle}{\langle a, a \rangle} a, & \text{if } a \neq 0,
\end{cases}
\]

\( Q_a = I - P_a \), the projection on a orthogonal complement of \([a]\), and \( s_a = (1 - |a|^2)^{1/2} \).

For \( p \in (0, \infty) \), we introduce the \( \mathcal{N}_p \)-space in \( \mathbb{B} \), which is defined as follows:

\[
\mathcal{N}_p(\mathbb{B}) := \left\{ f \in \mathcal{O}(\mathbb{B}) : \| f \|_p = \sup_{z \in \mathbb{B}} \left( \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \right)^{1/2} < \infty \right\},
\]

where \( dV \) is the Lebesgue normalized volume measure on \( \mathbb{B} \) (i.e. \( V(\mathbb{B}) = 1 \)).

It should be noted that in the case when \( p = 1 \), composition operators (respectively, weighted-composition operators) acting on the \( \mathcal{N}_p \)-space in the unit disk were considered in [4] (respectively, in [7]). Some results on boundedness and compactness of these operators were obtained.

The aim of the present note is to characterize the \( \mathcal{N}_p \)-spaces in the unit ball as well as the behavior of the weighted composition operators acting on these spaces. We study different properties of the weighted-composition operators and obtain the main results of [4,7] as particular cases.

2. Basic properties of \( \mathcal{N}_p \)-spaces in the ball

First we note that the automorphisms \( \Phi_a(z) \) have important properties, which we list here for the reader’s convenience. These properties are used in the proof of the main results of this note.

(i) \( \Phi_a(0) = a; \Phi_a(a) = 0. \)

(ii) \( \Phi_a'(0) = -s_a^2 P_a - s_a Q_a; \Phi_a'(a) = -\frac{P_a}{s_a^2} - \frac{Q_a}{s_a}. \)

(iii) The identity:

\[
1 - \langle \Phi_a(z), \Phi_a(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}
\]

holds for any \( z \in \mathbb{B}, w \in \mathbb{B}. \)

(iv) The identity:

\[
1 - |\Phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2}
\]

holds for any \( z \in \mathbb{B}. \)

(v) \( \Phi_a \) is an involution: \( \Phi_a(\Phi_a(z)) = z. \)

(vi) \( \Phi_a \) is a homeomorphism of the closed unit ball \( \overline{\mathbb{B}} \) onto itself.

It is clear that for \( n = 1 \), we have \( P_a = I \) and \( Q_a = 0 \), and hence \( \Phi_a(z) \) becomes the automorphism \( \sigma_a \) of the unit disk.

One of the most important results of this note is the following theorem, which gives various properties of \( \mathcal{N}_p \)-spaces. We note that among the statements in the theorem, assertion (a) plays a crucial role in the sequel.

**Theorem 2.1.** The following statements hold:

(a) For \( p > q > 0 \), we have \( H^\infty(\mathbb{B}) \hookrightarrow \mathcal{N}_q(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B}) \hookrightarrow A^{-\frac{n+1}{2}}(\mathbb{B}). \)

(b) For \( p > 0 \), if \( p > 2k - 1, k \in (0, \frac{n+1}{2}] \), then \( A^{-k}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B}). \) In particular, when \( p > n \), \( \mathcal{N}_p(\mathbb{B}) = A^{-\frac{n+1}{2}}(\mathbb{B}). \)
That is, $\mathcal{N}_p(B)$ is a functional Banach space with the norm $\| \cdot \|_p$, and moreover, its norm topology is stronger than the compact-open topology.

(d) For $0 < p < \infty$, $\mathcal{B}(B) \hookrightarrow \mathcal{N}_p(B)$, where $\mathcal{B}(B)$ is the Bloch space in $B$.

Note that the results of [4, Proposition 3.1] for the unit disk are contained in assertions (a), (b) and (c), respectively, when $n = 1$.

**Sketch of the proof of Theorem 2.1.** (a) The first two embeddings are easy. For the third one, denote $B_{1/2} = \{ z : |z| < \frac{1}{2} \}$. If $f \in \mathcal{N}_p(B)$, then:

$$\| f \|_p^2 \geq \left( \frac{3}{4} \right)^p \int_{B_{1/2}} |f(z)|^2 dV(z).$$

Furthermore, since $f \in \mathcal{O}(B)$, $|f|^2$ is subharmonic in $B_{2n}$, and hence by [3, Theorem 2.1.4 (8)], we have:

$$|f(0)|^2 \leq \frac{1}{V(B_{1/2})} \int_{B_{1/2}} |f(z)|^2 dV(z) = 4^n \int_{B_{1/2}} |f(z)|^2 dV(z). \quad (2.1)$$

Thus,

$$|f(0)|^2 \leq \frac{4p+n}{3p} \| f \|_p^2, \quad f \in \mathcal{N}_p(B). \quad (2.2)$$

For every fixed $z \in B$, we put

$$F_{z,f}(w) = (f \circ \Phi_z(w)) \cdot \frac{(1 - |z|^2)^{n+1}}{(1 - (w,z)^2)^{n+1}}, \quad w \in B,$$

which is clearly a holomorphic function in $B$. We can prove that $\| F_{z,f} \|_p^2 \leq \| f \|_p^2$, and so $F_{z,f} \in \mathcal{N}_p(B)$. Then, by (2.2), we have:

$$|f(z)|^2 (1 - |z|^2)^{n+1} = |F_{z,f}(0)|^2 \leq \frac{4p+n}{3p} \| F_{z,f} \|_p^2 \leq \frac{4p+n}{3p} \| f \|_p^2, \quad \forall z \in B,$$

which implies that:

$$|f|_{z+1}^2 = \sup_{z \in B} |f(z)| (1 - |z|^2)^{\frac{n+1}{2}} \leq \left( \frac{2p+n}{3p/2} \right)^{\frac{n+1}{2}} \| f \|_p, \quad \forall f \in \mathcal{N}_p(B). \quad (2.3)$$

That is, $\mathcal{N}_p(B) \hookrightarrow A^{-\frac{n+1}{2}}(B)$.

(b) Suppose $p > \max\{0, 2k - 1\}$, where $k \in (0, \frac{n+1}{2}]$. Let $f \in A^{-k}(B)$, we have:

$$\| f \|_p^2 \leq \| f \|_{k}^2 \sup_{a \in B} (1 - |a|^2)^p \int_{B} \frac{(1 - |z|^2)^{p-2k}}{|1 - (z,a)|^{2p}} dV(z).$$

Moreover, by [5, Proposition 1.4.10], we have for some positive constant $C$:

$$(1 - |a|^2)^p \int_{B} \frac{(1 - |z|^2)^{p-2k}}{|1 - (z,a)|^{2p}} dV(z) \leq C, \quad \forall a \in B,$$

which implies that

$$\| f \|_p \leq \sqrt{C} \| f \|_{k}, \quad \forall f \in \mathcal{N}_p(B).$$

That is $A^{-k}(B) \hookrightarrow \mathcal{N}_p(B)$.

(c) and (d) are easy. □
3. Weighted composition operators between $\mathcal{N}_p(B)$ and $A^{-q}(B)$

The following "probe" functions in $\mathcal{N}_p$-spaces are important in the proof of our main results in this section.

Lemma 3.1. For each $w \in B$, put:

$$k_w(z) := \left( \frac{1 - |w|^2}{(1 - (z, w))^2} \right)^{n+1}, \quad z \in B.$$ 

Then $k_w \in \mathcal{N}_p(B)$ and $\sup_{w \in B} \|k_w\|_p \leq 1$.

3.1. Boundedness

Note that the norm topology of both $\mathcal{N}_p(B)$ and $A^{-q}(B)$ is stronger than the compact-open topology, and hence it is stronger than the pointwise convergence topology. Thus, if the weighted-composition operator $W_{u, \varphi}$ maps $\mathcal{N}_p(B)$ into $A^{-q}(B)$, an application of closed graph theorem shows that $W_{u, \varphi}$ is automatically bounded from $\mathcal{N}_p(B)$ into $A^{-q}(B)$.

Theorem 3.2. Let $\varphi : B \rightarrow B$ be a holomorphic mapping, $u : B \rightarrow \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u, \varphi} : \mathcal{N}_p(B) \rightarrow A^{-q}(B)$ is bounded if and only if:

$$\sup_{z \in B} |u(z)| \left( \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} \right) < \infty. \quad (3.1)$$

Note when $n = 1$, Theorem 3.2 contains Theorem 3 in [7] as a particular case.

Sketch of the proof of Theorem 3.2. If $W_{u, \varphi} : \mathcal{N}_p(B) \rightarrow A^{-q}(B)$ is bounded, then by Lemma 3.1, there exists a positive constant $M$ such that:

$$M \geq |W_{u, \varphi}(k_{\varphi(z)}(z))|_q \geq |u(z)| \left( \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} \right), \quad \forall z \in B,$$

from which (3.1) follows.

Conversely, from (3.1), by Theorem 2.1(a), it follows that for some positive constant $M$:

$$|W_{u, \varphi}(f)|_q \leq \sup_{z \in B} |u(z)| \left( \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} \right) \cdot \|f\|_p \leq M \|f\|_p,$$

which shows that $W_{u, \varphi}$ is bounded from $\mathcal{N}_p(B)$ into $A^{-q}(B)$. $\square$

3.2. Compactness

By analogous arguments as of [1, Proposition 3.11], we can obtain the following test for compactness of $W_{u, \varphi}$.

Lemma 3.3. Let $\varphi : B \rightarrow B$ be a holomorphic mapping, $u : B \rightarrow \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u, \varphi} : \mathcal{N}_p(B) \rightarrow A^{-q}(B)$ is compact if and only if $|W_{u, \varphi}(f_m)|_q \rightarrow 0$, as $m \rightarrow \infty$ for any bounded sequence $\{f_m\}$ in $\mathcal{N}_p(B)$ which converges to 0 uniformly on every compact subset of $B$.

We have the following result.

Theorem 3.4. Let $\varphi : B \rightarrow B$ be a holomorphic mapping, $u : B \rightarrow \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u, \varphi} : \mathcal{N}_p(B) \rightarrow A^{-q}(B)$ is compact if and only if:

$$\lim_{t \rightarrow 1} \sup_{|\varphi(z)| > r} |u(z)| \left( \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} \right) = 0. \quad (3.2)$$

When $n = 1$, Theorem 3.4 contains Corollary 2 in [7] as a particular case.

Sketch of the proof of Theorem 3.4. If $W_{u, \varphi} : \mathcal{N}_p(B) \rightarrow A^{-q}(B)$ is compact, then it is bounded. By Theorem 3.2, $M = \sup_{z \in B} |u(z)| \left( \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} \right) < \infty$. 
Note that \( \lim_{t \to 1^-} F(t) \), where:
\[
F(t) = \sup_{|\varphi(z)| > t} |u(z)| \left( \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} \right).
\]
always exists. We show that (3.2) holds.

Assume on the contrary that \( \lim_{t \to 1^-} F(t) = L > 0 \). There exists an \( r_0 \in (0, 1) \) such that for all \( r \in (r_0, 1) \), we have \( F(r) > L/2 \). Then by the standard diagonal process, we can construct a sequence \( \{z_m\} \subset \mathbb{B} \) such that \( |\varphi(z_m)| \to 1 \) as \( m \to \infty \), and also for each \( m \in \mathbb{N} \), \( |u(z_m)| \cdot \frac{(1 - |z_m|^2)^q}{(1 - |\varphi(z_m)|^2)^{\frac{n+1}{2}}} \geq L/4 \). Consider the probe functions \( k_{w_m} \), where \( w_m = \varphi(z_m) \), defined in Lemma 3.1. It is easy to see that \( k_{w_m} \to 0 \) uniformly on every compact subset of \( \mathbb{B} \). Moreover, for each \( m \in \mathbb{N} \), \( \|k_{w_m}\|_p \leq 1 \).

Since \( W_{u, \varphi} \) is compact, by Lemma 3.3, \( |W_{u, \varphi}(k_{w_m})|_q \to 0 \) as \( m \to \infty \). However, for each \( m \in \mathbb{N} \),
\[
|W_{u, \varphi}(k_{w_m})|_q \geq |u(z_m)| \cdot |k_{w_m}(\varphi(z_m))|(1 - |z_m|^2)^q = |u(z_m)| \cdot \frac{(1 - |z_m|^2)^q}{(1 - |\varphi(z_m)|^2)^{\frac{n+1}{2}}} \geq \frac{L}{4}.
\]
which gives a contradiction.

Conversely, if (3.2) holds, and \( \{f_m\} \) is a bounded sequence in \( \mathcal{N}_p(\mathbb{B}) \) which converges to zero uniformly on every compact subset of \( \mathbb{B} \), then for a given \( \varepsilon > 0 \), there exist \( r_0 \in (0, 1) \) and \( m_\varepsilon \in \mathbb{N} \), such that for \( r \in (r_0, 1) \) and \( m > m_\varepsilon \), we have:
\[
|W_{u, \varphi}(f_m)|_q \leq \sup_{|\varphi(z)| > r} |u(z)| |f_m(\varphi(z))|(1 - |z|^2)^q + \sup_{|\varphi(z)| \leq r} |u(z)| |f_m(\varphi(z))|(1 - |z|^2)^q < \varepsilon.
\]
From this it follows that \( |W_{u, \varphi}(f_m)|_q \to 0 \) as \( m \to \infty \). By Lemma 3.3, we get the desired result. \( \square \)

As a corollary of Theorems 3.2 and 3.4, we have:

**Corollary 3.5.** Let \( \varphi : \mathbb{B} \to \mathbb{B} \) be a holomorphic mapping and \( p, q > 0 \). The composition operator \( C_\varphi \) acting from \( \mathcal{N}_p(\mathbb{B}) \to A^q(\mathbb{B}) \)

1. is bounded if and only if
   \[
   \sup_{z \in \mathbb{B}} \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty,
   \]
2. is compact if and only if
   \[
   \lim_{r \to 1^-} \sup_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} = 0.
   \]

When \( n = 1 \), Corollary 3.5 contains Theorems 4.1 and 4.3 in [4] as particular cases.

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**References**