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A remark on homomorphisms from right-angled Artin groups to mapping class groups ☆

Une remarque sur des homomorphismes entre les groupes d'Artin à angles droits et les groupes modulaires

Javier Aramayona [a], Juan Souto [b]

[a] *School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland*
[b] *Department of Mathematics, University of British Columbia, Vancouver, Canada*

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ABSTRACT

We study rigidity properties of certain homomorphisms from right-angled Artin groups to mapping class groups. As an application, we show that if $\Gamma \subset \mathrm{Map}(S)$ is a subgroup that contains some power of every Dehn twist, then any injective homomorphism $\Gamma \to \mathrm{Map}(S)$ is a restriction of an automorphism of $\mathrm{Map}(S)$.

RÉSUMÉ

Nous examinons la rigidité de certains homomorphismes entre groupes d'Artin rectangulaires et groupes modulaires. Nous démontrons que, si $\Gamma \subset \mathrm{Map}(S)$ est un sous-groupe qui contient quelque puissance de tout twist de Dehn, alors tout homomorphisme injectif $\Gamma \to \mathrm{Map}(S)$ est la restriction d'un automorphisme de $\mathrm{Map}(S)$.

Version française abrégée

Soit S une surface connexe et orientable, de genre g et avec n pointes. Nous supposerons que S est *non exceptionnelle*, c'est-à-dire que $3g+n \geqslant 5$ et $(g,n) \neq (1,2)$. Le *groupe modulaire étendu* $\mathrm{Map}^*(S)$ est le groupe de difféomorphismes de S à isotopie près. Le *groupe modulaire* $\mathrm{Map}(S)$ est le sous-groupe d'indice 2 en $\mathrm{Map}^*(S)$ dont les éléments sont représentés par les difféomorphismes préservant l'orientation de S. Finalement, le *groupe modulaire pur* $P\mathrm{Map}(S) \subset \mathrm{Map}(S)$ est le sous-groupe des éléments de $\mathrm{Map}(S)$ qui fixent chaque pointe de S.

Le groupe d'Artin rectangulaire $\mathbb{A}(X)$ associé à un complexe simplicial X est le groupe engendré par l'ensemble $X^{(0)}$ des sommets de X, et tel que les éléments correspondant à deux sommets voisins commutent. On remarque que, si $\Delta \subset X$ est un simplex, alors $\mathbb{A}(\Delta)$ est un sous-groupe de $\mathbb{A}(X)$ isomorphe à $\mathbb{Z}^{\dim(\Delta)+1}$.

Dans cet article, on s'intéresse aux homomorphismes *faiblement injectifs* $\rho : \mathbb{A}(X) \to \mathrm{Map}(S)$, où X est un sous-complexe *rigide* [1] du complexe des courbes $\mathcal{C}(S)$ de S [14]. Ici, nous disons que $X \subset \mathcal{C}(S)$ est rigide si toute application injective et simpliciale $\omega : X \to \mathcal{C}(S)$ est la restriction d'un automorphisme de $\mathcal{C}(S)$. Un homomorphisme ρ est faiblement injectif si,

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E-mail addresses: javier.aramayona@nuigalway.ie (J. Aramayona), jsouto@math.ubc.ca (J. Souto).

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pour tous les simplexes $\Delta, \Delta' \subset X$ et pour tous les éléments $\gamma \in \mathbb{A}(\Delta)$, $\gamma' \in \mathbb{A}(\Delta')$, si $\rho(\gamma) = \rho(\gamma')$, alors $\gamma = \gamma'$. Dénotant par δ_γ le twist de Dehn le long de $\gamma \in \mathcal{C}(S)$, on montre :

Théorème 1. *Soit S unesurface connexe, orientable et non exceptionnelle. Soit aussi $X \subset \mathcal{C}(S)$ un sous-complexe rigide avec* $\dim(X) = \dim(\mathcal{C})$*, et tel que chaque simplexe de X est l'intersection des simplexes de dimension maximale de X qui le contiennent. Pour tout homomorphisme faiblement injectif $\rho : \mathbb{A}(X) \to \mathrm{Map}(S)$, il existe $f \in \mathrm{Map}^*(S)$ et des fonctions $a, b : X^{(0)} \to \mathbb{Z} \setminus \{0\}$ telles que $\rho(\gamma^{a(\gamma)}) = f\delta_\gamma^{b(\gamma)} f^{-1}$ pout tout $\gamma \in X^{(0)}$. De plus, f est unique si S n'est pas la surface fermée de genre 2.*

Nous remarquons qu'il y a des constructions d'ensembles rigides *finis* de $\mathcal{C}(S)$ [1] qui satisfont les hypothèses du Théorème 1. Par exemple, soit $\mathbb{X}_n$ le complexe simplicial dont les simplexes de dimension k correspondent aux ensembles de $k+1$ diagonales disjointes du polygone avec n sommets. Pour $n \geqslant 5$, $\mathbb{X}_n$ est un sous-complexe rigide du complexe de courbes de la sphère $S_{0,n}$ avec n pointes [1]. Donc, si $\rho_0 : \mathbb{A}(\mathbb{X}_n) \to P\,\mathrm{Map}(S_{0,n})$ est l'homomorphisme faiblement injectif donné par $\rho_0(\gamma) = \delta_\gamma$ pour tout γ, alors tout homomorphisme injectif $\rho : \mathbb{A}(\mathbb{X}_n) \to P\,\mathrm{Map}(S_{0,n})$ a la forme $\rho(\cdot) = f((\rho_{\mathbb{X}_n} \circ \tau)(\cdot))f^{-1}$, où $f \in \mathrm{Map}^*(S_{0,n})$ et $\tau : \mathbb{A}(\mathbb{X}_n) \to \mathbb{A}(\mathbb{X}_n)$ est un monomorphisme.

Par ailleurs, le complexe des courbes lui-même est rigide [22]. En lui appliquant le Théorème 1, on montre :

Corollaire 2. *Soit S une surface connexe, orientable et non exceptionnelle, autre que la surface fermée de genre 2. Soit aussi $\Gamma \subset \mathrm{Map}(S)$ un sous-groupe tel que, pour toute $\gamma \in \mathcal{C}(S)$, il y a $n(\gamma) \in \mathbb{N}$ avec $\delta_\gamma^{n(\gamma)} \in \Gamma$. Pour tout homomorphisme injectif $\sigma : \Gamma \to \mathrm{Map}(S)$, il existe un unique élément $f \in \mathrm{Map}^*(S)$ tel que $\sigma(g) = fgf^{-1}$ pour tout $g \in \Gamma$.*

Rappelons que si S a pour genre au moins 3, les noyaux des homomorphismes du groupe modulaire dans des groupes de Lie compacts – par exemple les représentations quantiques [21] – satisfont la condition du Corollaire 2 [3].

On observe aussi que le Corollaire 2 implique que le groupe Γ est cohopfien et que son commensurateur abstrait est isomorphe à $\mathrm{Map}^*(S)$. Ces résultats sont déjà connus pour le group de Torelli [11] et pour quelques autres sous-groupes normaux de $\mathrm{Map}(S)$ d'indice infini [8,9]. Toutefois, la rigidité qu'on trouve dans le Corollaire 2 est bien plus forte que ce qu'on connaît dans ces cas-là : ici on ne suppose pas que $\sigma(\Gamma) \subset \Gamma$, ni que Γ et $\sigma(\Gamma)$ soient commensurables.

1. Introduction

Let S be a connected, orientable surface of genus g with n punctures. Throughout this note, we will assume that S is *non-exceptional*, that is, $3g+n \geqslant 5$ and $(g,n) \neq (1,2)$. Denote by $\mathrm{Map}^*(S)$ the *extended mapping class group*, that is the group of isotopy classes of self-diffeomorphisms of S. The *mapping class group* $\mathrm{Map}(S) \subset \mathrm{Map}^*(S)$ is the index 2 subgroup consisting of isotopy classes of those diffeomorphisms that preserve the orientation of S; finally, the *pure mapping class group* $P\,\mathrm{Map}(S) \subset \mathrm{Map}(S)$ is the subgroup of those mapping classes fixing each puncture of S.

Given a simplicial complex X, the *right-angled Artin group* $\mathbb{A}(X)$ associated with X is the group generated by the set $X^{(0)}$ of vertices of X, which verifies that $\gamma_i, \gamma_j \in X^{(0)}$ commute if and only if they are adjacent in X. Note that every simplex Δ of X determines an Abelian subgroup $\mathbb{A}(\Delta)$ of $\mathbb{A}(X)$, isomorphic to $\mathbb{Z}^{\dim(\Delta)+1}$.

In this note, we are interested in homomorphisms from right-angled Artin groups to mapping class groups. We remark that there are numerous examples of such homomorphisms: for instance, if $X \neq \emptyset$, then $\mathbb{A}(X)$ surjects onto $\mathbb{Z}$, and hence we obtain infinitely many homomorphisms $\mathbb{A}(X) \to \mathrm{Map}(S)$. In addition, so long as X has at least two non-adjacent vertices, $\mathbb{A}(X)$ surjects onto the non-Abelian free group of rank 2 and thus we obtain still more homomorphisms $\mathbb{A}(X) \to \mathrm{Map}(S)$. Observe, however, that the homomorphisms just described fail to be injective. On the other hand, Koberda [18] and Clay, Leininger and Mangahas [10] showed that *every* finitely generated right-angled Artin group embeds as a subgroup of some mapping class group.

Below, we will prove a rigidity result for a certain class of homomorphisms $\mathbb{A}(X) \to \mathrm{Map}(S)$, called *weakly injective*, in the case when X is a *rigid* subset of the curve complex $\mathcal{C}(S)$. We need a couple of definitions before stating our main result:

Definition *(Weak injectivity).* Let X be a simplicial complex, and G a group. A homomorphism $\rho : \mathbb{A}(X) \to G$ is *weakly injective* if the following holds: for all simplices $\Delta, \Delta' \subset X$, and for all $\gamma \in \mathbb{A}(\Delta)$, $\gamma' \in \mathbb{A}(\Delta')$, if $\rho(\gamma) = \rho(\gamma')$ then $\gamma = \gamma'$.

Recall that the curve complex $\mathcal{C}(S)$ is the simplicial complex whose k-simplices correspond to sets of $k+1$ distinct free isotopy classes of essential simple closed curves on S with pairwise disjoint representatives [14]. Denote by δ_γ the right Dehn twist along the simple closed curve $\gamma \in \mathcal{C}(S)$. If $X \subset \mathcal{C}(S)$ is an arbitrary subcomplex, then the homomorphism:

$$\rho_0 : \mathbb{A}(X) \to \mathrm{Map}(S), \qquad \rho_0(\gamma) = \delta_\gamma \quad \text{for every vertex } \gamma \in X^{(0)} \tag{1}$$

is weakly injective; see [12], in particular Section 3.3, for basic facts about Dehn twists. Note, however, that the map ρ_0 is not injective in general; compare with [13]. As mentioned earlier, we will be interested in subcomplexes of $\mathcal{C}(S)$ that are *rigid*.

Definition *(Rigid subcomplex).* A simplicial subcomplex X of $\mathcal{C}(S)$ is *rigid* if, for every injective simplicial map $\omega : X \to \mathcal{C}(S)$, there is an automorphism $\phi \in \mathrm{Aut}(\mathcal{C}(S))$ of $\mathcal{C}(S)$ with $\omega(\gamma) = \phi(\gamma)$ for all $\gamma \in X^{(0)}$.

Since S is assumed to be non-exceptional, the combination of results of Ivanov [17], Korkmaz [19], and Luo [20], implies that every automorphism of $\mathcal{C}(S)$ is induced by an element of $\mathrm{Map}^*(S)$. Furthermore, if S is not the closed surface of genus 2, then the said element is unique – see [20].

We are finally ready to state our main result:

Theorem 1. *Let S be a connected, orientable, and non-exceptional surface. Suppose that X is a rigid subcomplex of $\mathcal{C}(S)$, with $\dim(X) = \dim(\mathcal{C}(S))$, and such that every simplex of X is equal to the intersection of all maximal dimensional simplices of X that contain it. For every weakly injective homomorphism $\rho : \mathbb{A}(X) \to \mathrm{Map}(S)$, there are $f \in \mathrm{Map}^*(S)$ and functions $a, b : X^{(0)} \to \mathbb{Z} \setminus \{0\}$ with $\rho(\gamma^{a(\gamma)}) = f\delta_\gamma^{b(\gamma)} f^{-1}$, for every $\gamma \in X^{(0)}$. Moreover, f is unique unless S is a closed surface of genus 2.*

The equality $\rho(\gamma^{a(\gamma)}) = f\delta_\gamma^{b(\gamma)} f^{-1}$ asserts that $\rho(\gamma)$ is a root of a power of the Dehn twist along $f(\gamma)$. In the absence of roots – for instance if ρ takes values is the pure mapping class group $P\,\mathrm{Map}(S_{0,n})$ of the n-punctured sphere – we deduce that $\rho(\gamma)$ is in fact a power of a Dehn twist; with the notation of Theorem 1 this means that $a(\gamma) = 1$.

Concrete examples of finite rigid subsets of $\mathcal{C}(S_{0,n})$ were given in [1]. Indeed, the simplicial complex $\mathbb{X}_n$, whose k-simplices correspond to sets of $k+1$ pairwise disjoint diagonals of the polygon with n vertices, is a rigid subcomplex of $\mathcal{C}(S_{0,n})$ for $n \geqslant 5$. The complex $\mathbb{X}_n$ is the dual polytope to the associahedron, and hence every simplex is equal to the intersection of all maximal dimensional simplices containing it. We thus deduce from Theorem 1 that, for $n \geqslant 5$, every weakly injective homomorphism $\rho : \mathbb{A}(\mathbb{X}_n) \to P\,\mathrm{Map}(S_{0,n})$ is of the form $\rho(\cdot) = f((\rho_0 \circ \tau)(\cdot))f^{-1}$, where $f \in \mathrm{Map}^*(S_{0,n})$, ρ_0 is as in (1), and $\tau : \mathbb{A}(\mathbb{X}_n) \to \mathbb{A}(\mathbb{X}_n)$ is the injective homomorphism determined by $\tau(\gamma) = \gamma^{b(\gamma)}$ for $\gamma \in \mathbb{X}_n^{(0)}$.

We stress that in Theorem 1 we are *not* assuming that the subcomplex X be finite. In particular, applying the theorem to $X = \mathcal{C}(S)$, which is itself rigid by the work of Shackleton [22], we prove:

Corollary 2. *Let S be a connected, orientable, and non-exceptional surface, other than the closed surface of genus 2. Let $\Gamma \subset \mathrm{Map}(S)$ be a subgroup such that for every $\gamma \in \mathcal{C}(S)$ there is $n(\gamma) \in \mathbb{N}$ with $\delta_\gamma^{n(\gamma)} \in \Gamma$. For every injective homomorphism $\sigma : \Gamma \to \mathrm{Map}(S)$, there is a unique $f \in \mathrm{Map}^*(S)$ such that $\sigma(g) = fgf^{-1}$, for all $g \in \Gamma$.*

Since any finite index subgroup $\Gamma \subset \mathrm{Map}(S)$ automatically satisfies the hypothesis above, Corollary 2 implies the results in [4,5,15,16,22] about injections of finite index subgroups of mapping class groups.

In addition, there are numerous subgroups $\Gamma \subset \mathrm{Map}(S)$ of infinite index that satisfy the condition of Corollary 2, for example the kernel of any representation of $\mathrm{Map}(S)$ to a compact Lie group, provided that S has genus at least 3 – see Corollary 2.6 of [3]. This applies for instance to the so-called *quantum representations* [21] of $\mathrm{Map}(S)$, many of which have infinite image. Note also that Corollary 2 implies that the subgroup Γ is co-Hopfian, and that its abstract commensurator is isomorphic to $\mathrm{Map}^*(S)$. Such results were already known for the Torelli group [11], as well as for other infinite index normal subgroups of $\mathrm{Map}(S)$ [8,9]. However, we remark that the rigidity statement in Corollary 2 is more powerful than any of the existing ones, since we do not assume that $\sigma(\Gamma) \subset \Gamma$, or that Γ and $\sigma(\Gamma)$ are commensurable.

2. Abelian subgroups of the mapping class group

We recall a few standard facts about Abelian subgroups of the mapping class group. See [12] for basic facts on the mapping class group and [7] for details on its Abelian subgroups.

Let S be a connected orientable surface, and A an Abelian subgroup of $\mathrm{Map}(S)$. By the *rank* of A, we understand the dimension of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-vector space. A *reducing system* for A is an A-invariant multicurve $\lambda \subset S$. If there is no reducing system for A, then A contains a pseudo-Anosov and hence $\mathrm{rank}(A) = 1$. Thus, every Abelian subgroup A of $\mathrm{Map}(S)$ with $\mathrm{rank}(A) \geqslant 2$ is reducible. Given any reducing system λ for A, we have the exact sequence:

$$1 \to A \cap \mathbb{T}_\lambda \to A \to \mathrm{Map}(S \setminus \lambda), \tag{2}$$

where $\mathbb{T}_\lambda$ is the group generated by the Dehn twists along the components of λ (or half-twists in the case when the given component bounds a twice-punctured disk or a once-punctured torus). We say that λ is a *complete reducing system* for A if, for every component W of $S \setminus \lambda$, either:

(a) there are $d > 0$ and f in the image of the third homomorphism in (2), such that $f^d(W) = W$ and $f^d|_W$ is pseudo-Anosov, or
(b) there is $d > 0$ with $f^d|_W = \mathrm{Id}$ for every f in the image of the third homomorphism in (2).

There is a unique complete reducing system $\lambda(A)$ for A, the *canonical reducing system* [7], contained in every other complete reducing system for A. The *active surface* $S(A)$ of A is the union of those components of $S \setminus \lambda(A)$ for which (a) above is satisfied. Noting that no component of $S(A)$ is homeomorphic to a three-times punctured sphere, the pigeonhole principle and (2) together imply:

Fact 1. Suppose that S has genus g and n punctures. Every Abelian subgroup A of $\mathrm{Map}(S)$ satisfies $\mathrm{rank}(A) \leqslant 3g+n-3 = \dim(\mathcal{C}(S))+1$. Moreover, in the equality case, one has:

(1) $3g+n-3 = l(A)+s(A)$, where $l(A)$ and $s(A)$ are, respectively, the number of components of $\lambda(A)$ and $S(A)$;
(2) every component of $S(A)$ is homeomorphic to either a once-punctured torus or a four-times punctured sphere;
(3) the group A does not permute the components of $\lambda(A)$ (resp. $S(A)$).

Suppose now that $f \in \mathrm{Map}(S)$ has infinite order and is contained in some Abelian subgroup $A < \mathrm{Map}(S)$ of maximal rank. In particular, f is not pseudo-Anosov and thus $\lambda(\langle f\rangle) \neq \emptyset$. Moreover, observe that $\lambda(\langle f\rangle) \subset \lambda(A)$, and that $S(\langle f\rangle)$ is a union of connected components of $S(A)$. Moreover, both $\lambda(\langle f\rangle)$ and $S(\langle f\rangle)$ are preserved by the centralizer $\mathcal{Z}_{\mathrm{Map}(S)}(f)$ of f. In fact, the subgroup of $\mathcal{Z}_{\mathrm{Map}(S)}(f)$ that preserves each component of $\lambda(f)$ and each component of $S \setminus \lambda(f)$ has finite index in $\mathcal{Z}_{\mathrm{Map}(S)}(f)$ and contains $\langle f, \mathbb{T}_{\lambda(f)}\rangle$ in its center. Notice that $\mathrm{rank}(\langle f, \mathbb{T}_{\lambda(f)}\rangle) \geqslant 2$ unless $S(\langle f\rangle) = \emptyset$ and $\lambda(f)$ has a single component. Altogether we have:

Fact 2. Let S be a connected, orientable and non-exceptional surface. Suppose that $f \in \mathrm{Map}(S)$ has infinite order and is contained in an Abelian group of maximal rank. Then, either f is a root of a power of a Dehn twist, or the centralizer of f in $\mathrm{Map}(S)$ has a finite index subgroup G whose center $\mathcal{Z}(G)$ satisfies $\mathrm{rank}(\mathcal{Z}(G)) \geqslant 2$.

We can now prove:

Lemma 3. *Let S be a connected, orientable, and non-exceptional surface. If $\{A_i\}_{i\in I}$ is a collection of maximal rank Abelian subgroups of* $\mathrm{Map}(S)$ *such that* $\mathrm{rank}(\bigcap A_i) = 1$, *then $\bigcap A_i$ is a cyclic group generated by a root of a power of a Dehn twist.*

Proof. First, it follows from Fact 1 and the work of Birman and Hilden [6] (see Theorem 2.8 of [2] for an explicit statement) that the centralizer of a non-trivial finite-order element of $\mathrm{Map}(S)$ does not contain Abelian groups of maximal rank. Therefore $\bigcap A_i$ is torsion free, and hence cyclic.

Let G be a finite index subgroup of the centralizer of $\bigcap A_i$. Maximality of the rank of A_i implies that $\mathcal{Z}(G) \cap A_i$ has finite index in $\mathcal{Z}(G)$ for all i. In particular, also $\mathcal{Z}(G) \cap (\bigcap A_i)$ has finite index in $\mathcal{Z}(G)$; this proves that $\mathcal{Z}(G)$ has rank 1. By Fact 2, $\bigcap A_i$ is generated by a root of a power of a Dehn twist, as claimed.

A remark on roots. We remark that if f is a half-twist along a curve that bounds a twice-punctured disk or a once-punctured torus in S, then f is indeed contained in a maximal rank Abelian subgroup of $\mathrm{Map}(S)$. In fact, it is not difficult, albeit not so interesting and slightly cumbersome, to prove that these are the only roots which can appear in Lemma 3 as long as $(g,n) \neq (2,0)$. It follows that Theorem 1 can be marginally improved to assert that $a(\gamma) \in \{1,2\}$.

3. Proofs

Before proving the results announced in the introduction, we need a preparatory observation:

Lemma 4. *Let S be connected, orientable and non-exceptional surface. Suppose that X is a simplicial complex with* $\dim(X) = \dim(\mathcal{C}(S))$, *and whose every simplex is equal to the intersection of the maximal dimensional simplices of X that contain it. Then every weakly injective homomorphism $\rho : \mathbb{A}(X) \to \mathrm{Map}(S)$ maps each standard generator of $\mathbb{A}(X)$ to a root of a power of a Dehn twist along a single curve.*

Proof. Let $\gamma \in X^{(0)}$ be a vertex, and consider the collection $\{\Delta_i\}_{i\in I}$ of maximal dimensional simplices of X that contain γ. Our assumption implies that the cyclic group $\langle\gamma\rangle$ is equal to $\bigcap_i \mathbb{A}(\Delta_i)$. Since ρ is weakly injective, $\rho(\langle\gamma\rangle)$ is also an infinite cyclic subgroup of $\mathrm{Map}(S)$, which moreover satisfies:

$$\rho\big(\langle\gamma\rangle\big) = \bigcap_i \rho\big(\mathbb{A}(\Delta_i)\big) \subset \mathrm{Map}(S).$$

Now, $\mathrm{rank}(\rho(\mathbb{A}(\Delta_i))) = \mathrm{rank}(\mathbb{A}(\Delta_i)) = \dim(X)+1 = \dim(\mathcal{C}(S))+1$. We can hence apply Lemma 3 to $\{\rho(\Delta_i)\}_{i\in I}$, thus deducing that $\rho(\langle\gamma\rangle)$ is generated by a root of a power of a Dehn twist, as we needed to prove. □

We are now ready to prove Theorem 1:

Proof of Theorem 1. By Lemma 4, $\rho(\gamma)$ is a root of a power of a Dehn twist along a single curve, for every $\gamma \in X^{(0)}$. In other words, there are $\rho_*(\gamma) \in \mathcal{C}(S)$ and $a(\gamma), b(\gamma) \in \mathbb{Z} \setminus \{0\}$, with:

$$\rho\big(\gamma^{a(\gamma)}\big) = \delta_{\rho_*(\gamma)}^{b(\gamma)}.$$

Since the elements of $\mathbb{A}(X)$ corresponding to adjacent vertices $\gamma, \eta \in X^{(0)}$ commute, $\rho_*(\gamma)$ and $\rho_*(\eta)$ do not intersect. Moreover, if γ, η are arbitrary distinct vertices of X, then $\rho_*(\gamma) \neq \rho_*(\eta)$ because ρ is weakly injective. Therefore, we deduce that the map $\rho_* : X \to \mathcal{C}(S)$ is an injective simplicial map. Since $X \subset \mathcal{C}(S)$ is assumed to be rigid, there is $\phi \in \mathrm{Aut}(\mathcal{C}(S))$ with $\rho_*(\gamma) = \phi(\gamma)$ for all $\gamma \in X$. As S is not exceptional, the aforementioned results of Ivanov [17], Korkmaz [19] and Luo [20] together imply that there is $f \in \mathrm{Map}^*(S)$ with $\phi(\gamma) = f(\gamma)$ for all $\gamma \in \mathcal{C}(S)$; moreover, f is unique unless S is a closed surface of genus 2. Therefore, we obtain:

$$\rho\big(\gamma^{a(\gamma)}\big) = \delta_{\rho_*(\gamma)}^{b(\gamma)} = \delta_{f(\gamma)}^{b(\gamma)} = f\delta_\gamma^{b(\gamma)} f^{-1}$$

for all $\gamma \in X^{(0)}$, as desired. $\square$

Finally, we prove Corollary 2:

Proof of Corollary 2. Let $\sigma : \Gamma \to \mathrm{Map}(S)$ be an injective homomorphism, and $\rho : \mathbb{A}(\mathcal{C}(S)) \to \mathrm{Map}(S)$ the homomorphism $\rho(\gamma) = \delta_\gamma^{n(\gamma)}$, noting that its image is contained in Γ. Hence, we can also consider the homomorphism:

$$\rho' = \sigma \circ \rho : \mathbb{A}\big(\mathcal{C}(S)\big) \to \mathrm{Map}(S).$$

As S is assumed to be non-exceptional, $\mathcal{C}(S)$ is rigid [22] and thus Theorem 1 implies that there are $f \in \mathrm{Map}^*(S)$ and functions $a, b : \mathcal{C}(S) \to \mathbb{Z} \setminus \{0\}$ with $\rho'(\gamma^{a(\gamma)}) = f\delta_\gamma^{b(\gamma)} f^{-1}$ for every $\gamma \in \mathcal{C}(S)$. Moreover, f is unique since S is not the closed surface of genus 2. Conjugating σ by f^{-1}, we may in fact assume that:

$$\rho'\big(\gamma^{a(\gamma)}\big) = \delta_\gamma^{b(\gamma)}$$

for every vertex $\gamma \in \mathcal{C}(S)$. After this normalization, σ maps roots of powers of Dehn twists along a curve to roots of powers of Dehn twists along the same curve. We claim that $\sigma(h) = h$ for every $h \in \Gamma$. Indeed, note that for every $h \in \Gamma$ and $\gamma \in \mathcal{C}(S)$ there are $a, b, c \in \mathbb{Z}$ such that:

$$\delta_{h(\gamma)}^a = \sigma\big(\delta_{h(\gamma)}^b\big) = \sigma\big(h\delta_\gamma^b h^{-1}\big) = \sigma(h)\sigma\big(\delta_\gamma^b\big)\sigma(h)^{-1} = \sigma(h)\delta_\gamma^c\sigma(h)^{-1} = \delta_{\sigma(h)(\gamma)}^c.$$

This proves, in particular, that $h(\gamma) = \sigma(h)(\gamma)$. Since $\gamma \in \mathcal{C}(S)$ was arbitrary and S is not the closed surface of genus 2, it follows that $\sigma(h) = h$, as we needed to prove. $\square$

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