Algebraic Geometry

Surfaces in \( \mathbb{P}^4 \) whose 4-secant lines do not sweep out a hypersurface

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**Abstract**

We prove that a smooth surface in \( \mathbb{P}^4 \) whose 4-secant lines do not sweep out a hypersurface of \( \mathbb{P}^4 \) either lies on a pencil of cubic hypersurfaces, or else is linked to a Veronese surface by the complete intersection of a cubic and a quartic hypersurface.

**Résumé**

Nous montrons qu'une surface lisse dans \( \mathbb{P}^4 \) dont les droites quadrisécantes ne couvrent pas une hypersurface de \( \mathbb{P}^4 \) est, soit contenue dans un pinceau de cubiques, soit liée à une surface de Veronese via l'intersection complète d'une cubique et d'une quartique.

1. Introduction

Let \( X \subset \mathbb{P}^4 \) be a smooth complex projective surface. A line \( L \subset \mathbb{P}^4 \) is said to be \( k \)-secant to \( X \) if \( X \cap L \) is a finite scheme of length at least \( k \). While the 2-secant lines of \( X \) fill up \( \mathbb{P}^4 \) unless \( X \) lies on a hyperplane, Aure [2] characterized the elliptic quintic scrolls — refining earlier work of Severi in his celebrated paper [19] — as the only smooth surfaces not lying on a quadric hypersurface whose 3-secant lines do not fill up \( \mathbb{P}^4 \), as conjectured by Peskine. On the other hand, Ran's generalization of the classical Trisecant Lemma [18] shows that the 4-secant lines of \( X \) never fill up \( \mathbb{P}^4 \). In this case, \( X \) is expected to have a 2-dimensional family of 4-secant lines sweeping out a hypersurface of \( \mathbb{P}^4 \). Therefore, it is natural to ask whether there are any exceptions to this expected behavior. Of course, the 4-secant lines of a surface lying on a pencil of cubic hypersurfaces do not sweep out a hypersurface, so in the spirit of Aure's work we show that a smooth surface whose 4-secant lines do not sweep out a hypersurface of \( \mathbb{P}^4 \) either lies on a pencil of cubic hypersurfaces, or else is linked to a Veronese surface by the complete intersection of a cubic and a quartic hypersurface. We would like to emphasize the analogy with Aure's result, which in fact can be rephrased by saying that a smooth surface whose 3-secant lines do not fill up \( \mathbb{P}^4 \) either lies on a quadric hypersurface, or else is linked to a Veronese surface by the complete intersection of two cubic hypersurfaces.

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In higher dimensions, Ran [17] proved — under an extra assumption that is satisfied as soon as \( n > 4 \) — that the \((n + 1)\)-secant lines of a smooth \( n \)-dimensional subvariety \( X \subset \mathbb{P}^{n+2} \) fill up the ambient space if \( X \) does not lie on a hypersurface of degree \( n \). On the other hand, Mezzetti [15, Theorem 0.2] and Kwak [10, Theorem 3.4(b)] obtained some partial results that suggest that the same could be true in the case \( n = 3 \). In view of [18] and our result, it would be interesting to study also the smooth \( n \)-dimensional subvarieties of \( \mathbb{P}^{n+2} \) whose \((n + 2)\)-secant lines do not sweep out a hypersurface of \( \mathbb{P}^{n+2} \) (cf. [10, Open questions 4.7]), but we will not address this problem here.

Going back to the case \( n = 2 \), there are several ways to proceed. In this paper, we give a short proof based on Le Barz’s formula [13] for the 4-secant cycle of \( X \subset \mathbb{P}^4 \), that allows us to express the Euler characteristic \( \chi(O_X) \) in terms of the degree \( d \) and the sectional genus \( g \) of \( X \). Now we come to the key fact of the proof: as the 4-secant lines of \( X \) do not sweep out a hypersurface of \( \mathbb{P}^4 \), the inner projection from a general point of \( X \) into \( \mathbb{P}^3 \) does not have any triple point, and hence we can express \( g \) in terms of \( d \) thanks to Kleiman’s triple-point formula. To conclude the proof, Halphen’s bound yields a short list of admissible pairs \((d, g)\) for which the corresponding surface is well known.

We point out that Bauer [3] classified — in response to a conjecture of Van de Ven — the smooth surfaces \( X \subset \mathbb{P}^5 \) whose 3-secant lines do not sweep out a 3-dimensional subvariety of \( \mathbb{P}^3 \) in a similar way, that is, using Le Barz’s formula for the 3-secant cycle of \( X \subset \mathbb{P}^3 \) and noting that the inner projection from a general point of \( X \) into \( \mathbb{P}^3 \) does not have any double point.

Finally, we mention that smooth surfaces with no 4-secant lines were classified first by Bertolini and Turrini [4], as explained in Remark 4.

2. Proof

We work over the field of complex numbers.

**Theorem.** Let \( X \subset \mathbb{P}^4 \) be a smooth surface whose 4-secant lines do not sweep out a hypersurface of \( \mathbb{P}^4 \). Then either \( X \) lies on a pencil of cubic hypersurfaces, or else \( X \) is linked to a Veronese surface by the complete intersection of a cubic and a quartic hypersurface.

The proof is based on the following formula. Let \( d \) denote the degree of \( X \), let \( g := g(C) \) denote the genus of a general hyperplane section \( C \) of \( X \), and let \( \chi := \chi(O_X) \) denote the Euler characteristic of \( X \).

**Le Barz’s formula.** (See [13] and [14].) The number \( N_4 \) of 4-secant lines of a smooth surface \( X \subset \mathbb{P}^4 \) meeting a general line, if finite, is:

\[
N_4 = \frac{1}{8} \left( d^4 - 10d^3 + 7d^2 + d \left( 35 - 8g \right) + 2d \left( 28g - 33 \right) + 4 \left( g^2 - 25g + 24 \right) + 8 \chi \left( 2d - 9 \right) \right).
\]

The key fact of the proof is the following:

**Lemma.** If the 4-secant lines of a smooth surface \( X \subset \mathbb{P}^4 \) do not sweep out a hypersurface and \( X \) is not a scroll (i.e. \( X \) is not covered by lines), then

\[
g = \frac{1}{6} \left( 9d - 33 \pm \sqrt{\Delta(d)} \right),
\]

where \( \Delta(d) := 3d^4 - 72d^3 + 636d^2 - 2448d + 3465 \).

**Proof.** Let \( x \in X \) be a general point, and let \( \text{Bl}_x(X) \) denote the blowing-up of \( X \) at \( x \). It follows from the hypotheses that the map \( f : \text{Bl}_x(X) \to \mathbb{P}^3 \) induced by the inner projection \( \pi_x : X \to \mathbb{P}^3 \) is finite and does not have any triple point. Hence we apply Kleiman’s triple-point formula to \( f \) (see [9] for the general picture; see also [13] for our particular situation), so

\[
\chi = \frac{1}{12} \left( -d^3 + 9d^2 - 2d(16 - 3g) - 12(2g - 5) \right)
\]

(cf. [6, Proposition 3.2]) and the statement follows from Le Barz’s formula since \( N_4 = 0 \). \( \square \)

**Remark 1.** On the other hand, if \( X \subset \mathbb{P}^4 \) is a scroll then there exists a smooth irreducible curve \( B \subset \mathbb{G}(1, 4) \) of genus \( g(B) \) such that \( X \cong \mathbb{P}(E) \), where \( E \) denotes the rank-2 universal bundle on \( \mathbb{G}(1, 4) \) restricted to \( B \). Then \( g = g(B) \), \( \chi = 1 - g \), \( K^2 = 8 - 8g \) and hence \( g = \left( d^2 - 5d + 6 \right) / 6 \) by the well-known double-point formula

\[
d^2 = 5d + 10(g - 1) + 2K^2 - 12\chi.
\]

Therefore, if \( N_4 = 0 \) then \((d, g) \in \{(2, 0), (3, 0), (5, 1)\}\) (cf. [11] and [1]).
Proof of the theorem. If $X \subset \mathbb{P}^4$ is a scroll then $(d, g) \in \{(2,0), (3,0), (5,1)\}$ by Remark 1. Otherwise, it follows from the lemma that $g = (9d - 33 + \sqrt{\Delta(d)})/6$. If $g = (9d - 33 - \sqrt{\Delta(d)})/6$ then $\Delta(d)$ bound yields $d < 10$ and hence $(d, g) \in \{(3,1), (4,1), (5,2), (6,4), (7,5), (8,6), (9,6)\}$. On the other hand, if $g = (9d - 33 + \sqrt{\Delta(d)})/6$ then Halphen’s bound yields $d < 20$ and hence $(d, g) \in \{(3,1), (4,1), (5,2), (6,4), (7,5), (8,7), (9,10)\}$. If $(d, g) = (9,6)$ then $\chi = -4$, so $X$ would be a ruled surface, and hence $K^2 = -31$ by the double-point formula. This contradicts the inequality $K^2 \leq 8\chi$. The rest of the cases are effective, and $X$ is well known in all of them. As $g$ is maximal (in the sense of [7]) except in the cases $(d, g) \in \{(5,1), (6,3), (8,6), (9,6)\}$, a simple description of $X$ and $\mathcal{I}_X$ follows by linkage. Moreover, if $(d, g) = (6,3)$ then $X$ is linked to a cubic scroll by a complete intersection $(3,3)$. If $(d, g) = (4,0)$ then $h^1(\mathcal{I}_X(1)) = 1$, and hence $X$ is a projected Veronese surface by Severi’s theorem [19]. Finally, in the cases $(d, g) \in \{(5,1), (8,6)\}$ one can easily describe $X$ as a surface linked to a Veronese surface by a complete intersection $(3,3)$ and $(3,4)$, respectively.

Remark 2. Surfaces cut out by cubic hypersurfaces do not have any 4-secant line. Let us describe the family of 4-secant lines in the cases in which $X \subset \mathbb{P}^4$ is not cut out by cubic hypersurfaces, namely $(d, g) \in \{(8,7), (8,6)\}$:

(i) If $X$ is linked to a plane $X'$ by a c.i. $(3,3)$, then it has a resolution:

$$0 \to O_{\mathbb{P}^4}(−1)^{62} \to O_{\mathbb{P}^4} \oplus O_{\mathbb{P}^4}(1)^{62} \to \mathcal{I}_X(4) \to 0.$$ 

In this case, $X$ is a minimal elliptic surface over $\mathbb{P}^1$ with Kodaira dimension $\kappa = 1$ (see [16] or [8]). It has a unique plane quartic curve $P \subset X'$, and it is fibered by the pencil $|H − P|$ of elliptic quartic curves.

(ii) If $X$ is linked to a Veronese surface by a c.i. $(3,4)$ then it has a resolution:

$$0 \to T_{\mathbb{P}^4}(−2) \to O_{\mathbb{P}^4}^{66} \oplus O_{\mathbb{P}^4}(1) \to \mathcal{I}_X(4) \to 0.$$ 

In this case $σ : X \to \mathbb{P}^2$ is the blowing-up along 16 points $\{x_1, \ldots, x_4, y_1, \ldots, y_{12}\}$ lying on a quartic of $\mathbb{P}^2$ and embedded in $\mathbb{P}^4$ by the linear system $|σ^*(4L − \sum x_i + \sum y_j)|$ and $|σ^*(5L − x_i + \sum y_j)|$, and it is ruled by five pencils of rational quartic curves, namely $|σ^*(2L − x_i)|$ and $|σ^*(L − x_i)|$.

Remark 3. As expected, one can check that the Cayley–Le Barz formula (see [5] and [12]):

$$\frac{1}{12} (d - 2)(d - 3)^2(d - 4) - \frac{1}{2} g (d^2 - 7d + 13 - g)$$

for the number, if finite, of 4-secant lines of $C \subset \mathbb{P}^3$ gives 1 in the case (i), where $(d, g) = (8,7)$, and 5 in the case (ii), where $(d, g) = (8,6)$.

Remark 4. If the family of 4-secant lines of a smooth surface $X \subset \mathbb{P}^4$ is at most 1-dimensional, then $C$ does not have any 4-secant line, so the Cayley–Le Barz formula and Halphen’s bound yield

$$(d, g) \in \{(2,0), (3,0), (3,1), (4,0), (4,1), (5,1), (5,2), (6,3), (6,4), (7,5), (9,10)\}$$

and hence $X$ is cut out by cubic hypersurfaces (cf. [4]).

References


