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Weighted moments for the limit of a normalized supercritical Galton–Watson process

Moments pondérés pour la limite d'un processus de Galton–Watson normalisé supercritique

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ABSTRACT

Let (Z_n) be a supercritical Galton–Watson process, and let W be the limit of the normalized population size Z_n/m^n, where $m = \mathbb{E}Z_1 > 1$ is the mean of the offspring distribution. Let ℓ be a positive function slowly varying at ∞. Bingham and Doney (1974) [4] showed that for $\alpha > 1$ not an integer, $\mathbb{E}W^\alpha \ell(W) < \infty$ if and only if $\mathbb{E}Z_1^\alpha \ell(Z_1) < \infty$; Alsmeyer and Rösler (2004) [2] proved the equivalence for $\alpha > 1$ not a dyadic power. Here we prove it for all $\alpha > 1$.

RÉSUMÉ

Soient (Z_n) un processus de Galton–Watson surcritique et W la limite de la population normalisée Z_n/m^n, où $m = \mathbb{E}Z_1 > 1$ est la moyenne de la loi de reproduction. Soit ℓ une fonction positive à variation lente en ∞. Bingham et Doney (1974) [4] ont montré que, pour $\alpha > 0$ non entier, $\mathbb{E}W^\alpha \ell(W) < \infty$ si et seulement si $\mathbb{E}Z_1^\alpha \ell(Z_1) < \infty$; Alsmeyer et Rösler (2004) [2] ont montré l'équivalence lorsque $\alpha > 1$ n'est pas une puissance de 2. Nous le montrons ici pour tout $\alpha > 1$.

Version française abrégée

Soit $(Z_n)_{n\geqslant 0}$ un processus de Galton–Watson défini sur un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$, avec $Z_0 = 1$, de loi de reproduction $\{p_i\}_{i\in\mathbb{N}}$, où $p_i \geqslant 0$, $\sum_{i\in\mathbb{N}} p_i = 1$, $\mathbb{N} = \{0, 1, 2, \ldots\}$. Par définition,

$$Z_0 = 1 \quad \text{et} \quad Z_{n+1} = \sum_{u\in T_n} X_u \quad \text{pour } n \geqslant 0, \tag{0.1}$$

où tous les X_u, indexés par des suites d'entiers u, sont des variables aléatoires à valeurs entières, de loi commune $\{p_i\}_{i\in\mathbb{N}}$, et mutuellement indépendantes. Ici, T_n désigne l'ensemble des individus de la n^{e} génération, représentés par des suites

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u d'entiers positifs de longueur $|u| = n$: comme d'habitude, la particule initiale est représentée par la suite nulle $\emptyset$ (de longueur 0) ; si $u \in T_n$, alors $ui \in T_{n+1}$ si et seulement si $1 \leqslant i \leqslant X_u$. Nous considérons le cas surcritique où :

$$m := \sum_{i=0}^{\infty} i p_i > 1.$$

Il est bien connu que la suite :

$$W_n = \frac{Z_n}{m^n} \quad (n \geqslant 0)$$

forme une martingale non négative par rapport à la filtration naturelle :

$$\mathcal{E}_0 = \{\emptyset, \Omega\} \quad \text{et} \quad \mathcal{E}_n = \sigma\{X_u \colon |u| < n\} \quad \text{pour } n \geqslant 1.$$

Soit

$$W = \lim_{n\to\infty} W_n \quad \text{et} \quad W^* = \sup_{n\geqslant 0} W_n,$$

où la limite existe presque sûrement par le théorème de convergence des martingales, et $\mathbb{E}W \leqslant 1$ par le lemme de Fatou. Par le théorème de Kesten–Stigum, $\mathbb{E}W = 1$ si $\mathbb{E}Z_1 \log^+ Z_1 < \infty$, et $\mathbb{P}(W = 0) = 1$ sinon, où $\log^+ x = \log x$ si $x \geqslant 1$ et $\log^+ x = 0$ si $0 \leqslant x < 1$.

Lorsque $\mathbb{P}(W = 0) < 1$, l'existence des moments de W est une question intéressante, qui a été étudiée par de nombreux auteurs ; voir par example Harris [8], Athreya et Ney [3], Bingham et Doney [4], Alsmeyer et Rösler [2]. De théorèmes de comparaison sur les moments pondérés de W et de Z_1 ont un intérêt particulier. Soit ℓ une fonction positive à variation lente en ∞. Par un théorème Tauberien profond, Bingham et Doney [4] ont d'abord mentré que, lorsque $\alpha > 1$ n'est pas un entier, $\mathbb{E}W^\alpha \ell(W) < \infty$ si et seulement si $\mathbb{E}Z_1^\alpha \ell(Z_1) < \infty$. Par un joli argument de martingale, Alsmeyer et Rösler [2] ont montré ensuite que l'équivalence reste vraie lorsque α n'est pas de la forme 2^n pour un certain entier $n \geqslant 1$. Dans cette note, nous montrons que l'équivalence est toujours vraie dès que $\alpha > 1$, par un raffinement de l'argument de martingale d'Alsmeyer et Rösler [2]. Soit :

$$R_0 = \left\{\ell : [0,\infty) \to [0,\infty) \colon \ell \text{ est mesurable et } \lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \ \forall \lambda > 0\right\}$$

la classe des fonctions à variation lente en ∞. Voici le résultat principal de la note :

Théorème 0.1. *Soient $\alpha > 1$ et $\ell \in R_0$. Alors les assertions suivantes sont équivalentes* : (a) $\mathbb{E}Z_1^\alpha \ell(Z_1) < \infty$; (b) $\mathbb{E}W^{*\alpha} \ell(W^*) < \infty$; (c) $\mathbb{E}W = 1$ *et* $\mathbb{E}W^\alpha \ell(W) < \infty$.

1. Introduction and results

Let $(Z_n)_{n\geqslant 0}$ be a Galton–Watson process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $Z_0 = 1$ and offspring distribution $\{p_i\}_{i\in\mathbb{N}}$, where $p_i \geqslant 0$ and $\sum_{i=0}^{\infty} p_i = 1$. By definition,

$$Z_0 = 1 \quad \text{and} \quad Z_{n+1} = \sum_{u\in T_n} X_u \quad \text{for } n \geqslant 0, \tag{1.1}$$

where all X_u, indexed by finite sequences of integers u, are integer-valued random variables with common distribution $\{p_i\}_{i\in\mathbb{N}}$, and are independent of each other. Here T_n denotes the set of all particles of the nth generation, represented by sequences u of positive integers of length $|u| = n$: as usual, the initial particle is marked by the empty sequence $\emptyset$ (of length 0); if $u \in T_n$, then $ui \in T_n$ if and only if $1 \leqslant i \leqslant X_u$. We consider the supercritical case where:

$$m := \sum_{i=0}^{\infty} i p_i > 1.$$

It is well-known that the sequence:

$$W_n = \frac{Z_n}{m^n} \quad (n \geqslant 0)$$

forms a nonnegative martingale with respect to the natural filtration:

$$\mathcal{E}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{E}_n = \sigma\{X_u \colon |u| < n\} \quad \text{for } n \geqslant 1.$$

Let:

$$W = \lim_{n\to\infty} W_n \quad \text{and} \quad W^* := \sup_{n\geqslant 0} W_n,$$

where the limit exists almost surely by the martingale convergence theorem, and $\mathbb{E}W \leqslant 1$ by Fatou's lemma. The famous Kesten–Stigum theorem states that $\mathbb{E}W = 1$ if $\mathbb{E}Z_1 \log^+ Z_1 < \infty$, and $\mathbb{P}(W = 0) = 1$ otherwise, where $\log^+ x = \log x$ when $x \geqslant 1$ and $\log^+ x = 0$ when $0 \leqslant x < 1$.

When $\mathbb{P}(W = 0) < 1$, the existence of moments of W is an interesting problem that has been studied by many authors; see for example Harris [8], Athreya and Ney [3], Bingham and Doney [4], and Alsmeyer and Rösler [2]. Comparison theorems about weighted moments of W and Z_1 are of particular interest. Let ℓ be a positive function slowly varying at ∞. Via a deep Tauberian theorem, Bingham and Doney [4] showed that when $\alpha > 1$ is not an integer, $\mathbb{E}W^\alpha \ell(W) < \infty$ if and only if $EZ_1^\alpha \ell(Z_1) < \infty$. By a nice martingale argument, Alsmeyer and Rösler [2] showed that the equivalence remains true when α is not of the form 2^n for some integer $n \geqslant 1$. In this note, we show that the equivalence is always true whenever $\alpha > 1$, by a refinement of the martingale argument of Alsmeyer and Rösler [2]. Let:

$$R_0 = \left\{\ell : [0,\infty) \to [0,\infty),\ \ell \text{ is measurable and } \lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1\ \forall \lambda > 0\right\}$$

be the set of functions slowly varying at ∞. Below is the main result of the note.

Theorem 1.1. *Let $\alpha > 1$ and $\ell \in R_0$. Then the following assertions are equivalent*: (a) $\mathbb{E}Z_1^\alpha \ell(Z_1) < \infty$; (b) $\mathbb{E}W^{*\alpha}\ell(W^*) < \infty$; (c) $\mathbb{E}W = 1$ *and* $\mathbb{E}W^\alpha \ell(W) < \infty$.

We mention that the argument of this note can be adapted to show that the same result also holds for a branching process in an independent and identically distributed random environment and for weighted branching processes (in a deterministic or random environment), thus extending and completing the corresponding results in [10,7,14,13,9,11,1]: see [12]. Here we only consider the usual Galton–Watson process, to show rapidly the essential elements of the argument, which leads to the progress on this classical process.

2. Key idea of the approach

The proof of Theorem 1.1 is mainly based on a double martingale structure and convex inequalities for martingales, by a refinement of the martingale argument of Alsmeyer and Rösler [2].

For $n \geqslant 1$, define:

$$D_n = W_n - W_{n-1} = \frac{1}{m^{n-1}} \sum_{u\in T_{n-1}} \tilde{X}_u, \quad \text{where } \tilde{X}_u = \frac{X_u}{m} - 1. \tag{2.1}$$

Then $(D_n, \mathcal{E}_n)_{n\geqslant 1}$ forms a sequence of martingale differences, and $W^* = \sup_{n\geqslant 0} W_n$ can be written as:

$$W^* = 1 + \sup_{n\geqslant 1}(D_1 + \cdots + D_n). \tag{2.2}$$

For convenience, we will write $\mathbb{P}_n$ for the conditional probability of $\mathbb{P}$ given $\mathcal{E}_n$, and $\mathbb{E}_n$ for the corresponding expectation. Note that for all $n \geqslant 1$ and all sequences u with length n, $\tilde{X}_u$ are independent and identically distributed with $\mathbb{E}\tilde{X}_u = 0$. Therefore, under $\mathbb{P}_{n-1}$, D_n can also be considered as the sum of martingale difference sequences. Therefore W_n and D_n constitute a double martingale structure.

Using Burkholder–Davis–Gundy (BDG) inequalities (see [6, Chap. 11, Theorems 1 and 2]) both to (W_n) and to (D_n), we shall obtain the following key result that we will use in the proof of Theorem 1.1. We use 1_n to denote for the sequences of length n whose components are all equal to 1.

Theorem 2.1. *Let ϕ be a convex and increasing function with $\phi(0) = 0$ and $\phi(2x) \leqslant c\phi(x)$ for some constant $c \in (0,\infty)$ and all $x > 0$. Let $\beta \in (1,2]$. If the function $x \mapsto \phi(x^{1/\beta})$ is also convex, and $\mathbb{E}|\tilde{X}|^\beta < \infty$, then we have*:

$$\mathbb{E}\phi\big(W^* - 1\big) \leqslant C \sum_{n=1}^{\infty} \frac{\mathbb{E}\phi(W_{n-1}^{1/\beta})}{m^{(n-1)(\beta-1)}} + C \sum_{n=1}^{\infty} \mathbb{E}\phi\left(\frac{|\tilde{X}_{1_{n-1}}| \cdot W_{n-1}^{1/\beta}}{m^{(n-1)(\beta-1)/\beta}}\right), \tag{2.3}$$

where $C = C(\phi,\beta)$ is a constant depending only on ϕ and β.

Notice that $\mathbb{E}|\tilde{X}|^\beta < \infty$ if and only if $\mathbb{E}Z_1^\beta < \infty$. As an example of applications of Theorem 2.1, applying (2.3) with $\phi(t) = t^{\beta^k}$ for $k = 1, 2, \ldots$ and using an induction argument on k, we see that for each $\beta \in (0,2]$ and each $k = 1, 2, \ldots,$

$$\mathbb{E}(Z_1)^{\beta^k} < \infty \quad \Rightarrow \quad \mathbb{E}\big(W^*\big)^{\beta^k} < \infty. \tag{2.4}$$

As for each $\alpha \in (1,\infty)$, we can take an integer $k \geqslant 1$ large enough such that $\beta := \alpha^{1/k} \in (1,2]$, so the above implication shows that for each $\alpha \in (1,\infty)$,

$$\mathbb{E}(Z_1)^{\alpha} < \infty \quad \Rightarrow \quad \mathbb{E}\big(W^*\big)^{\alpha} < \infty. \tag{2.5}$$

This leads to the essential part of the following famous equivalences that we shall use later: for each $\alpha \in (1,\infty)$,

$$\mathbb{E}(Z_1)^{\alpha} < \infty \quad \Leftrightarrow \quad \mathbb{E}\big(W^*\big)^{\alpha} < \infty \quad \Leftrightarrow \quad \mathbb{E}W^{\alpha} < \infty. \tag{2.6}$$

Bingham and Doney [4] proved the equivalence $\mathbb{E}(Z_1)^{\alpha} < \infty \Leftrightarrow \mathbb{E}W^{\alpha} < \infty$ by a Tauberian theorem on Laplace transforms.

In our argument, we use BDG inequality instead of the Topchii–Vatutin inequality (see [15, Theorem 2]), which was used in Alsmeyer and Rösler [2], to avoid the condition that the derivative of ϕ is concave. Here we introduce β just to unify the proof for all $\alpha > 1$; in fact, when $\alpha > 2$, it suffices to choose $\beta = 2$, as in Alsmeyer and Rösler [2].

Proof of Theorem 2.1. Write $\phi_{1/\beta}(x) = \phi(x^{1/\beta})$ for $x \geqslant 0$. By (2.2) and the general form of the BDG inequality (cf. [6, Chap. 11, p. 427, Theorem 2]),

$$\mathbb{E}\phi\big(W^* - 1\big) \leqslant B\left(\mathbb{E}\phi_{1/\beta}\left(\sum_{n=1}^{\infty}\mathbb{E}_{n-1}|D_n|^{\beta}\right) + \sum_{n=1}^{\infty}\mathbb{E}\phi\big(|D_n|\big)\right), \tag{2.7}$$

where $B > 0$ is a constant depending only on c and β. By (2.1), under P_{n-1}, D_n is the sum of independent and centered random variables. Therefore by the usual BDG inequality (cf. [6, Chap. 11, p. 425, Theorem 1]) or the Marcinkiewicz–Zygmund inequality, together with the subadditivity of $x \mapsto x^{\beta/2}$, we obtain:

$$\mathbb{E}_{n-1}|D_n|^{\beta} \leqslant \frac{B}{m^{(n-1)\beta}}\mathbb{E}_{n-1}\left[\sum_{|u|=n-1}(\tilde{X}_u)^2\right]^{\beta/2} \leqslant \frac{B}{m^{(n-1)\beta}}\mathbb{E}_{n-1}\sum_{|u|=n-1}|\tilde{X}_u|^{\beta} = \frac{CW_{n-1}}{m^{(n-1)(\beta-1)}},$$

where $C = B\mathbb{E}|\tilde{X}|^{\beta}$. Hence, defining A by $\sum_{n=1}^{\infty}\frac{1}{Am^{(n-1)(\beta-1)}} = 1$ and using the convexity of $\phi_{1/\beta}$, we get:

$$\begin{aligned}\phi_{1/\beta}\left(\sum_{n=1}^{\infty}\mathbb{E}_{n-1}|D_n|^{\beta}\right) &\leqslant \phi_{1/\beta}\left(\sum_{n=1}^{\infty}\frac{1}{Am^{(n-1)(\beta-1)}}\cdot ACW_{n-1}\right) \leqslant \sum_{n=1}^{\infty}\frac{1}{Am^{(n-1)(\beta-1)}}\phi_{1/\beta}(ACW_{n-1})\\ &= \sum_{n=1}^{\infty}\frac{1}{Am^{(n-1)(\beta-1)}}\cdot\phi\big(C^{1/\beta}A^{1/\beta}W_{n-1}^{1/\beta}\big) \leqslant \sum_{n=1}^{\infty}\frac{C_1\phi(W_{n-1}^{1/\beta})}{Am^{(n-1)(\beta-1)}},\end{aligned} \tag{2.8}$$

where $C_1 > 0$ is a constant depending only on C and c. For the second part of (2.7), again by the usual BDG inequality, the fact that $(\sum_i x_i^2)^{1/2} \leqslant (\sum_i x_i^{\beta})^{1/\beta}$ for $x_i \geqslant 0$ (subadditivity of $x \mapsto x^{\beta/2}$), and the convexity of $\phi_{1/\beta}$, we have (using the identity $\sum_{|u|=n-1}\frac{1}{Z_{n-1}} = 1$):

$$\begin{aligned}\mathbb{E}_{n-1}\phi\big(|D_n|\big) &\leqslant B\mathbb{E}_{n-1}\phi_{1/\beta}\left(\sum_{|u|=n-1}\frac{|\tilde{X}_u|^{\beta}}{m^{\beta(n-1)}}\right) = B\mathbb{E}_{n-1}\phi_{1/\beta}\left(\sum_{|u|=n-1}\frac{1}{Z_{n-1}}\frac{Z_{n-1}|\tilde{X}_u|^{\beta}}{m^{\beta(n-1)}}\right)\\ &\leqslant B\mathbb{E}_{n-1}\sum_{|u|=n-1}\frac{1}{Z_{n-1}}\phi_{1/\beta}\left(\frac{Z_{n-1}|\tilde{X}_u|^{\beta}}{m^{\beta(n-1)}}\right) = B\mathbb{E}_{n-1}\phi\left(\frac{|\tilde{X}_{1_{n-1}}|}{m^{(n-1)(\beta-1)/\beta}}\cdot W_{n-1}^{1/\beta}\right).\end{aligned}$$

Therefore

$$\mathbb{E}\phi\big(|D_n|\big) \leqslant B\mathbb{E}\phi\left(\frac{|\tilde{X}_{1_{n-1}}|}{m^{(n-1)(\beta-1)/\beta}}W_{n-1}^{1/\beta}\right). \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we get (2.3). □

3. Sketch of the proof of Theorem 1.1

Proof of Theorem 1.1. Let $\beta \in (1,2]$ with $\beta < \alpha$. Write $\phi(x) = x^{\alpha}\ell(x)$. Without loss of generality, we can assume that both $\phi(x)$ and $\phi(x^{1/\beta})$ are increasing and convex on $[0,\infty)$ with $\ell(x) > 0$ for all $x \geqslant 0$.

(i) We first show that (a) implies (b). By Theorem 2.1, to show the finiteness of (b), we just need to show the finiteness of the two terms on the right side of (2.3).

Potter's theorem (see [5, Theorem 1.5.6]) implies that for any $\epsilon > 0$, $\ell(x) \leqslant C \max(x^\epsilon, x^{-\epsilon})$ for some $C > 0$ and all $x > 0$. Choose $0 < \epsilon \leqslant \alpha(\beta - 1)/(\beta + 1)$. Then:

$$\frac{\mathbb{E}\phi(W_{n-1}^{1/\beta})}{m^{(n-1)(\beta-1)}} \leqslant \frac{C}{m^{(n-1)(\beta-1)}} \left(\mathbb{E}W_{n-1}^{\frac{\alpha+\epsilon}{\beta}} + \mathbb{E}W_{n-1}^{\frac{\alpha-\epsilon}{\beta}}\right). \tag{3.1}$$

As $\mathbb{E}Z_1^{(\alpha+\epsilon)/\beta} \leqslant \mathbb{E}Z_1^{\alpha-\epsilon} \leqslant Cm^{\alpha-\epsilon}(1 + \mathbb{E}\phi_1(W_1)) < \infty$, it follows from (2.6) that $\mathbb{E}(W^*)^{\frac{\alpha+\epsilon}{\beta}} < \infty$. Similarly, $\mathbb{E}(W^*)^{\frac{\alpha-\epsilon}{\beta}} < \infty$. Since $m > 1$, we see that the left part of (3.1) is summable on n, which shows the finiteness of the first term on the right side of (2.3).

We now consider the second term on the right side of (2.3). Again by Potter's theorem, for all $\epsilon > 0$, there exists $C > 0$ such that $\ell(xy) \leqslant C\ell(x) \cdot \max\{y^\epsilon, y^{-\epsilon}\}$ for all $x > 0$ and $y > 0$. Noting that $\tilde{X}_{1_{n-1}}$ is independent of W_{n-1}, we have:

$$\mathbb{E}\phi\left(\frac{|\tilde{X}_{1_{n-1}}| \cdot W_{n-1}^{1/\beta}}{m^{(n-1)(\beta-1)/\beta}}\right) \leqslant C\mathbb{E}\phi\left(|\tilde{X}_{1_{n-1}}|\right)\left(\frac{\mathbb{E}W_{n-1}^{(\alpha+\epsilon)/\beta}}{m^{(n-1)(\beta-1)(\alpha+\epsilon)/\beta}} + \frac{\mathbb{E}W_{n-1}^{(\alpha-\epsilon)/\beta}}{m^{(n-1)(\beta-1)(\alpha-\epsilon)/\beta}}\right). \tag{3.2}$$

Since $\mathbb{E}\phi(|\tilde{X}_{1_{n-1}}|) = \mathbb{E}\phi(W_1 - 1) < \infty$, combining with the facts that $\mathbb{E}(W^*)^{\frac{\alpha+\epsilon}{\beta}} < \infty$ and $\mathbb{E}\phi(W^*)^{\frac{\alpha-\epsilon}{\beta}} < \infty$, we see that the left part of (3.2) is summable on n, which shows the finiteness of the second term on the right side of (2.3).

Therefore, we have $\mathbb{E}\phi(W^* - 1) < \infty$, which is equivalent to $\mathbb{E}\phi(W^*) < \infty$.

(ii) We then show that (b) implies (c). Assume (b). Since $W \leqslant W^*$, we have $\mathbb{E}\phi(W) \leqslant \mathbb{E}\phi(W^*) < \infty$; by the dominated convergence theorem, W_n converges to W in L^1, so that $\mathbb{E}W = 1$. Thus (c) holds.

(iii) We finally show that (c) implies (a). Notice that the limit W satisfies the distributional equation:

$$W = \sum_{i=1}^{Z_1} \frac{W^{(i)}}{m}, \tag{3.3}$$

where $(W^{(i)})$ are independent of each other and independent of Z_1, each $W^{(i)}$ has the same law as W. Hence, by Jensen's inequality:

$$\mathbb{E}\phi(W) \geqslant \mathbb{E}\phi\left(\mathbb{E}\left(\sum_{i=1}^{Z_1} \frac{W^{(i)}}{m} \middle| \mathcal{E}_1\right)\right) = \mathbb{E}\phi\left(\frac{Z_1}{m}\right) = \mathbb{E}\phi(W_1). \tag{3.4}$$

So we have $\mathbb{E}\phi(W_1) < \infty$, which is equivalent to $\mathbb{E}\phi(Z_1) < \infty$. $\square$

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