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A new relation between the condensation index of complex sequences and the null controllability of parabolic systems



Une nouvelle relation entre l'indice de condensation des suites complexes et la contrôlabilité à zéro des systèmes paraboliques

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ABSTRACT

In this note, we present a new result that relates the condensation index of a sequence of complex numbers with the null controllability of parabolic systems. We show that a minimal time is required for controllability. The results are used to prove the boundary controllability of some coupled parabolic equations.

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RÉSUMÉ

On annonce un résultat qui relie l'indice de condensation des suites complexes et la contrôlabilité à zéro des systèmes paraboliques. On montre qu'un temps minimal de contrôle est nécessaire. Ces résultats sont ensuite utilisés pour étudier la contrôlabilité à zéro par le bord de quelques systèmes paraboliques.

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1. Notation and main results

Let \mathbb{X} be a Hilbert space on \mathbb{C} with norm and inner product respectively denoted by $\|\cdot\|$ and (\cdot, \cdot) . Let us consider $\{\phi_k\}_{k \geq 1}$ a Riesz basis of \mathbb{X} and denote $\{\psi_k\}_{k \geq 1}$ the corresponding biorthogonal sequence to $\{\phi_k\}_{k \geq 1}$. Also consider a sequence $\Lambda = \{\lambda_k\}_{k \geq 1} \subset \mathbb{C}$, with $\lambda_i \neq \lambda_k$ for all $i \neq k$, satisfying for a $\delta > 0$,

$$\Re(\lambda_k) \geq \delta |\lambda_k| > 0, \quad \forall k \geq 1, \quad \text{and} \quad \sum_{k \geq 1} \frac{1}{|\lambda_k|} < \infty. \quad (1)$$

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Denote by \mathbb{X}_{-1} the completion of \mathbb{X} with respect to the norm: $\|y\|_{-1} := (\sum_{k \geq 1} \frac{|(y, \psi_k)|^2}{|\lambda_k|^2})^{1/2}$. Also the Hilbert space $(\mathbb{X}_1, \|\cdot\|_1)$ is defined by $\mathbb{X}_1 := \{y \in \mathbb{X} : \|y\|_1 < \infty\}$ with $\|y\|_1^2 = \sum_{k \geq 1} |\lambda_k|^2 |(y, \psi_k)|^2$. Furthermore, let $\mathcal{A} : \mathcal{D}(\mathcal{A}) = \mathbb{X}_1 \subset \mathbb{X} \rightarrow \mathbb{X}$ be the operator given by:

$$\mathcal{A} = - \sum_{k \geq 1} \lambda_k (\cdot, \psi_k) \phi_k. \tag{2}$$

Let us fix $T > 0$ a real number and $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ (so $\mathcal{B}^* \in \mathcal{L}((\mathbb{X}_{-1})', \mathbb{C}) \equiv \mathbb{X}_{-1}$). We consider:

$$y' = \mathcal{A}y + \mathcal{B}u \quad \text{on } (0, T); \quad y(0) = y_0 \in \mathbb{X}. \tag{3}$$

In System (3), $u \in L^2(0, T; \mathbb{C})$ is the control that acts on the system by means of the operator \mathcal{B} . We assume that \mathcal{B} is an admissible control operator for the semigroup generated by \mathcal{A} , i.e., for a positive time T^* one has $\mathcal{R}(L_{T^*}) \subset \mathbb{X}$, where $L_T u = \int_0^T e^{(T-s)\mathcal{A}} \mathcal{B}u(s) ds$. System (3) is approximately controllable in \mathbb{X} at time $T > 0$ if for every $y_0 \in \mathbb{X}$, $\mathcal{R}(T) = \{y(T) = e^{T\mathcal{A}}y_0 + L_T u \text{ with } u \in L^2(0, T; \mathbb{C})\}$ is dense in \mathbb{X} and System (3) is null controllable in \mathbb{X} at time $T > 0$ if for all $y_0 \in \mathbb{X}$, $0 \in \mathcal{R}(T)$. It is well known that the controllability properties of System (3) amount to appropriate properties of the so-called adjoint system to System (3). This adjoint system has the form:

$$-\varphi' = \mathcal{A}^* \varphi \quad \text{on } (0, T); \quad \varphi(T) = \varphi_0 \in \mathbb{X}. \tag{4}$$

Observe that, for any $\varphi_0 \in \mathbb{X}$, System (4) admits a unique weak solution $\varphi \in C^0([0, T]; \mathbb{X})$. Classical results (see e.g. [6, Theorem 11.2.1]) imply:

Theorem 1.1. Assume that $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ is an admissible control operator for the semigroup $\{e^{t\mathcal{A}}\}_{t>0}$ generated by \mathcal{A} , with \mathcal{A} given by (2), and $\Lambda = \{\lambda_k\}_{k \geq 1}$ is a complex sequence satisfying (1). Then, System (3) is approximately controllable in \mathbb{X} at time T if and only if:

$$b_k := \mathcal{B}^* \psi_k \neq 0, \quad \forall k \geq 1. \tag{5}$$

Moreover, (3) is null controllable in \mathbb{X} at time T if and only if there exists a constant $C_T > 0$ such that:

$$\sum_{k \geq 1} e^{-2T\Re(\lambda_k)} |a_k|^2 \leq C_T \int_0^T \left| \sum_{k \geq 1} \bar{b}_k e^{-\lambda_k(T-t)} a_k \right|^2, \quad \forall \{a_k\}_{k \geq 1} \in \ell^2(\mathbb{C}). \tag{6}$$

Our main result reads as follows:

Theorem 1.2. Assume that $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ is an admissible control operator for the semigroup $\{e^{t\mathcal{A}}\}_{t>0}$ and $\Lambda = \{\lambda_k\}_{k \geq 1}$ is a complex sequence satisfying respectively (5) and (1). For $z \in \mathbb{C}$, let us introduce $E(z) = \prod_{k=1}^{\infty} (1 - \frac{z^2}{\lambda_k^2})$ and $T_0 = \limsup (\frac{\log \frac{1}{|b_k|}}{\Re(\lambda_k)} + \frac{\log \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)})$. Then System (3) is null controllable for $T > T_0$ and is not null controllable for $T < T_0$.

The condensation index of a sequence $\Lambda = \{\lambda_k\}_{k \geq 1} \subset \mathbb{C}$ satisfying (1) is the real number $c(\Lambda) = \limsup \frac{\log \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)}$, where the function E is given in Theorem 1.2. The condensation index is related to the overconvergence of Dirichlet series (see [5]). Observe that when $\lim \frac{\log |b_k|}{\Re(\lambda_k)} = 0$, then, $T_0 = c(\Lambda)$.

2. Idea of the proof of Theorem 1.2

The proof is technical and long and the details are given in [2]. For the proof of the positive result, we transform the control problem into a problem of moments. So we need to study the existence of biorthogonal families to complex exponentials and study some properties of these families. We have the following result:

Theorem 2.1. Let $\Lambda = \{\lambda_k\}_{k \geq 1} \subset \mathbb{C}$ be a sequence satisfying (1) and fix $T \in (0, \infty]$. Let $A(\Lambda, T) = \overline{\text{span}\{e^{-\lambda_k t} : k \geq 1\}^{L^2(0, T; \mathbb{C})}}$. Then, there exists a biorthogonal family $\{q_k\}_{k \geq 1} \subset A(\Lambda, T)$ to $\{e^{-\lambda_k t}\}_{k \geq 1}$ such that for any $\varepsilon > 0$ one has:

$$C_{1,\varepsilon} \frac{e^{-\varepsilon \Re(\lambda_k)}}{|E'(\lambda_k)|} \leq \|q_k\|_{L^2(0, T; \mathbb{C})} \leq C_{2,\varepsilon} \frac{e^{\varepsilon \Re(\lambda_k)}}{|E'(\lambda_k)|}, \quad \forall k \geq 1, \tag{7}$$

where E is the function given in Theorem 1.2 and $C_{1,\varepsilon}, C_{2,\varepsilon} > 0$ are constants only depending on ε, Λ and T .

The null controllability problem for System (3) reduces to the following moment problem: find $u \in L^2(0, T; \mathbb{C})$ such that, for b_k given by (5), we have $\bar{b}_k \int_0^T e^{-\lambda_k t} u(T-t) dt = -e^{-\lambda_k T} (y_0, \psi_k), \forall k \geq 1$. We can solve this equality using the characterization of the biorthogonal family given above. So: $u(t) = v(T-t) = -\sum_{k \geq 1} \frac{e^{-\lambda_k T}}{b_k} (y_0, \psi_k) \bar{q}_k(T-t)$. It follows that if $T > T_0$, with T_0 given in Theorem 1.2, the previous series is absolutely convergent in $L^2(0, T; \mathbb{C})$ and thus $u \in L^2(0, T; \mathbb{C})$. Indeed, if we choose $\varepsilon \in (0, T - T_0)$, then (7) leads to: $\| \frac{e^{-\lambda_k T}}{b_k} (y_0, \psi_k) \bar{q}_k \|_{L^2(0, T; \mathbb{C})}^2 \leq C_\varepsilon e^{-2\Re(\lambda_k)(T-T_0-\varepsilon)} |(y_0, \psi_k)|^2, \forall k \geq k_\varepsilon \geq 1$.

We prove that System (3) is not null controllable at time T , when $T < T_0$, showing that inequality (6) does not hold. Without loss of generality, we can assume that the sequence $\Lambda = \{\lambda_k\}_{k \geq 1} \subset \mathbb{C}$ is normally ordered, i.e., $|\lambda_k| \leq |\lambda_{k+1}|$ for any $k \geq 1$ and $\arg(\lambda_k) < \arg(\lambda_{k+1})$ when $|\lambda_k| = |\lambda_{k+1}|$. The negative part of Theorem 1.2 is a consequence of the following result:

Theorem 2.2. *Let $\Lambda = \{\lambda_k\}_{k \geq 1} \subset \mathbb{C}$ be a normally ordered sequence satisfying condition (1). Then, there exists a sequence of sets $\Delta = \{G_k\}_{k \geq 1}$ such that $\bigcup_{k \geq 1} G_k \cap \Lambda = \Lambda$ and for any subsequence $\{\lambda_{n_k}\}_{k \geq 1} \subseteq \Lambda$, one has:*

$$\lim \left(\frac{\log \frac{1}{|E'(\lambda_{n_k})|}}{\Re(\lambda_{n_k})} - \frac{1}{\Re(\lambda_{n_k})} \log \left| \frac{q_k!}{P'_{D_k}(\lambda_{n_k})} \right| \right) = 0, \tag{8}$$

where $\{D_k\}_{k \geq 1} \subseteq \Delta$ is a subsequence of sets satisfying $\lambda_{n_k} \in D_k$ and $q_k + 1$ is the cardinal of the set $D_k \cap \Lambda$. In the previous equality, P_A is the polynomial function $P_A(z) = \prod_{\lambda \in \Lambda} (z - \lambda)$.

Suppose that the observability inequality (6) holds. Using the previous result, we introduce $a_n^{(k)} = \frac{p_k!}{b_n P'_{G_k}(\lambda_n)}$ if $\lambda_n \in G_k$ and 0 otherwise ($p_k + 1$ is the cardinal of $G_k \cap \Lambda$). Clearly, the (finite) sequence $\{a_n^{(k)}\}_{n \geq 1}$ lies in $\ell^2(\mathbb{C})$. From (6), we can write:

$$\sigma_k^{(1)} := \sum_{\lambda_n \in G_k} \left| \frac{p_k!}{b_n P'_{G_k}(\lambda_n)} e^{-\lambda_n T} \right|^2 \leq C_T \int_0^T \left| \sum_{\lambda_n \in G_k} \frac{p_k!}{P'_{G_k}(\lambda_n)} e^{-\lambda_n t} \right|^2 dt := \sigma_k^{(2)}, \quad \forall k \geq 1. \tag{9}$$

Using the Lebesgue Theorem, it can be shown that $\lim \sigma_k^{(2)} = 0$. On the other hand, from the definition of T_0 (see Theorem 1.2) and (8), there exists $\{n_k\}_{k \geq 1}$ such that $T_0 = \lim \frac{1}{\Re(\lambda_{n_k})} (\log |\frac{1}{b_{n_k}}| + \log |\frac{q_k!}{P'_{D_k}(\lambda_{n_k})}|)$, where $\{D_k\}_{k \geq 1} \subseteq \Delta$ is a subsequence of sets satisfying $\lambda_{n_k} \in D_k$, for any k , and $q_k + 1$ is the cardinal of the set $D_k \cap \Lambda$. Observe that $\sigma_{n_k}^{(1)} \geq \left| \frac{q_k!}{b_{n_k} P'_{D_k}(\lambda_{n_k})} e^{-\lambda_{n_k} T} \right|^2 = e^{2\Re(\lambda_{n_k})[\frac{1}{\Re(\lambda_{n_k})} (\log |\frac{1}{b_{n_k}}| + \log |\frac{q_k!}{P'_{D_k}(\lambda_{n_k})}|) - T]}$. This last inequality shows $\lim \sigma_{n_k}^{(1)} = \infty$. This contradicts (9). For details, see [2].

3. An application: A boundary controllability problem

For $T > 0$ and $Q = (0, \pi) \times (0, T)$, consider the one-dimensional controlled (non-scalar) system:

$$\left\{ \frac{\partial y}{\partial t} - \left(\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) y = 0, \quad \text{in } Q, \quad y(0, \cdot) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} v, \quad y(\pi, \cdot) = 0 \quad \text{on } (0, T), \tag{10}$$

and initial datum $y(\cdot, 0) = y_0$ in $(0, \pi)$, $y_0 \in H^{-1}(0, \pi; \mathbb{R}^2)$ and $d > 0$. Observe that $v \in L^2(0, T)$ is a scalar boundary control that acts on the Dirichlet boundary condition of the state at point $x = 0$ by means of the vector $(b_1, b_2)^\top$. The aim is to control the whole system (two states) with a control force v .

The control problem (10) has been completely solved in [3] when $d = 1$. For a general system of $n \geq 2$ coupled equations with $M = I_n$, see [1]. The controllability problem for System (10) when $d \neq 1$ is more intricate and only few results are known. For $b_1 = 0$ and $b_2 = 1$: Firstly, System (10) is approximately controllable in $H^{-1}(0, \pi; \mathbb{R}^2)$ at time T if and only if $\sqrt{d} \notin \mathbb{Q}$ (see [3]). Secondly, there exists $d \in (0, \infty)$ with $\sqrt{d} \notin \mathbb{Q}$ such that System (10) is not null controllable at any time $T > 0$ (see [4]).

To our knowledge and apart from the previous results, the controllability properties of System (10) are completely open in the case $d \neq 1$. As a consequence of Theorem 1.2, we have:

Theorem 3.1. *Assume $d \neq 1$ and let $c(\Lambda_d)$ be the condensation index of the sequence $\Lambda_d := \{k^2, dk^2\}_{k \geq 1}$. Then,*

- (i) System (10) is approximately controllable in $\mathbb{X} = H^{-1}(0, \pi; \mathbb{R}^2)$ at any time $T > 0$ if and only if $\sqrt{d} \notin \mathbb{Q}$ and $b_2[(d-1)k^2 b_1 + db_2] \neq 0$.
- (ii) System (10) is null controllable in \mathbb{X} at any time $T > c(\Lambda_d)$ and is not null controllable in \mathbb{X} for $T < c(\Lambda_d)$.
- (iii) For any $\tau_0 \in [0, \infty]$, there exists $d \in (0, \infty)$ with $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = \tau_0$.

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