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Algebraic geometry

On a family of complex algebraic surfaces of degree 3n

Sur une famille de surfaces algébriques complexes de degré 3n

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Article history: Received 21 January 2013 Accepted 17 September 2013 Available online 21 October 2013 Presented by Claire Voisin	We study a class of algebraic surfaces of degree $3n$ in the complex projective space with only ordinary double points. They are obtained by using bivariate polynomials with complex coefficients related to the generalized cosine associated with the affine Weyl group of the root system A_2 . © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É
	Nous étudions une classe de surfaces algébriques de degré $3n$ dans l'espace projectif complexe, avec seulement des points doubles ordinaires. Ils sont générés par des polynômes complexes qui sont liés au cosinus généralisé associé au groupe de Weyl affine du système de racines A_2 .

In [2], Chmutov introduced surfaces V_d of degree d in the complex projective space $\mathbf{P}^3(\mathbf{C})$ given by the affine equations $P_d(u, v) + T_d(w) = 0$, where P_ds are the generalized Chebyshev polynomials or folding polynomials [6,7] associated with the affine Weyl group $\tilde{W}(A_2)$ of the root system A_2 , and T_d is a Chebyshev polynomial. In this work, we study a class of surfaces of degree 3*n* that have more singularities than the Chmutov surfaces of the same degree.

For d = 3n, n = 1, 2, 3, ... the surfaces U_d are given by the affine equations:

$$Q_d(u, v) + T_d(w) = 0$$
 (1)

where T_d is the Chebyshev polynomial of degree d with two critical values 2 and 3, and Q_d is defined as follows. The generalized cosine associated with the affine Weil group $W(A_2)$ is $h(u, v) = (h_1, h_2)$, where:

$$h_1(u, v) := e^{-2\pi i u} + e^{-2\pi i v} + e^{2\pi i (u+v)}, \qquad h_2(u, v) := e^{2\pi i u} + e^{2\pi i v} + e^{-2\pi i (u+v)}$$
(2)

The polynomials P_d appearing in the surfaces V_d are defined in such a way that $P_d(h_1(u, v), h_2(u, v)) := C_d(u, v)$, where $C_d(u, v) := h_{1,d}(u, v) + h_{2,d}(u, v), h_{1,d}(u, v) = h_1(du, dv), h_{2,d}(u, v) = h_2(du, dv).$ For (x, y) := h(u, v), it can be shown [6,2] that $P_d(x, y) = j_d(x, y) + j_d(y, x)$, where $j_d(x, y)$ is the determinant of a *dxd* matrix:

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$$j_d(x, y) = \det \begin{pmatrix} x & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 2y & x & 1 & 0 & 0 & 0 & \dots & 0 \\ 3 & y & x & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & y & x & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & y & x & 1 & 0 \dots & 0 \\ \dots & 0 \\ \dots & 0 \\ \dots & \dots & \dots & 0 & 1 & y & x & 1 \\ 0 & \dots & \dots & 0 & 0 & 1 & y & x \end{pmatrix}$$

The number of non-degenerate singularities of V_d for d = 3n is [2]:

$$\binom{d}{2} \left\lfloor \frac{d}{2} \right\rfloor + \left(\frac{d^2}{3} - d \right) \left\lfloor \frac{d-1}{2} \right\rfloor$$
(3)

For the construction of the surfaces U_d , we use:

$$g_{1,d}(u,v) := e^{-2\pi i (du + \frac{2}{3})} + e^{-2\pi i (dv - \frac{1}{3})} + e^{2\pi i (d(u+v) + \frac{1}{3})}, \qquad g_{2,d} := e^{2\pi i (du + \frac{2}{3})} + e^{2\pi i (dv - \frac{1}{3})} + e^{-2\pi i (d(u+v) + \frac{1}{3})}$$
(4)

There are bivariate polynomials Q_d such that $Q_d(h_1(u, v), h_2(u, v)) := H_d(u, v)$, where:

$$H_d(u,v) := g_{1,d}(u,v) + g_{2,d}(u,v) = 2\cos\left(2\pi du - \frac{2\pi}{3}\right) + 2\cos\left(2\pi dv - \frac{2\pi}{3}\right) + 2\cos\left(2\pi d(u+v) + \frac{2\pi}{3}\right)$$
(5)

Having in mind that $g_{1,d}(u, v) = e^{\frac{2\pi i}{3}}h_{1,d}(u, v)$ and $g_{2,d}(u, v) = e^{-\frac{2\pi i}{3}}h_{2,d}(u, v)$, we get, for (x, y) := h(u, v):

$$Q_d(x, y) = e^{\frac{2\pi i}{3}} j_d(x, y) + e^{-\frac{2\pi i}{3}} j_d(y, x)$$
(6)

The homogeneous part of highest degree in $Q_d(x, y)$ is $e^{\frac{2\pi i}{3}}x^d + e^{-\frac{2\pi i}{3}}y^d$; therefore, Q_d has $(d-1)^2$ critical points.

Lemma. The polynomial Q_d has $\frac{d(d-3)}{6}$ critical points with critical value 6, $\binom{d}{2}$ critical points with critical value -2 and $\frac{d^2}{3} - d + 1$ critical points with critical value -3. All the critical points of Q_d are non-degenerate.

Proof. We restrict H_d and h on the plane with real coordinates. The Jacobi matrices satisfy $J(H_d) = J(Q_d) \circ J(h)$. Zeros of the Jacobian determinant of h:

$$\det J(h) = 4\pi^2 e^{-2\pi i(u+\nu)} \left(e^{-2\pi i\nu} - e^{-2\pi iu} \right) \left(e^{2\pi i(u+2\nu)} - 1 \right) \left(e^{2\pi i(2u+\nu)} - 1 \right)$$
(7)

are the sides of the triangle Δ whose interior is given by u - v > 0, u + 2v > 0, 2u + v < 1, which is the fundamental region of $\tilde{W}(A_2)$. All the points from one orbit of $\tilde{W}(A_2)$ are mapped by h into a single point, and the images of the critical points of H_d from the interior of Δ by h are the critical points of Q_d .

A direct computation of the critical points of H_d leads to three cases. In the following list, we indicate the critical value ζ and the number of points N_{ζ} corresponding to ζ inside Δ by (ζ, N_{ζ}) :

(a)
$$(6, \frac{d(d-3)}{6}); u = \frac{3k+1}{d}, v = \frac{3l+1}{d}; k, l \in \mathbb{Z}.$$

(b) $(-3, \frac{d^2}{3} - d + 1); u = \frac{k}{3d}, v = \frac{l}{3d}$, with $k = 3m - 1, l = 3p - 1$ or $k = 3m, l = 3p; m, p \in \mathbb{Z}.$
(b1) $(-3, 1 + \frac{d(d-3)}{6}); u = \frac{3m+2}{3d}, v = \frac{3p+2}{3d}; m, p \in \mathbb{Z}.$
(b2) $(-3, \frac{d(d-3)}{6}); u = \frac{m}{d}, v = \frac{p}{d}; m, p \in \mathbb{Z}.$
(c) $(-2, \binom{d}{2}); u = \frac{3k+2}{6d}, v = \frac{3l+2}{6d}$ with k or l odd.
(c1) $(-2, \frac{d(d-1)}{3}); u = \frac{6m-1}{6d}, v = \frac{3p-1}{6d}; m, p \in \mathbb{Z}.$

Z.

(c2)
$$(-2, \frac{d(d-1)}{6}); u = \frac{6m+2}{6d}, v = \frac{6p-1}{6d}; m, p \in$$

The hessian matrix of H_d :

$$\begin{pmatrix} -8\pi^2 d^2 (\cos(2\pi du - \frac{2\pi}{3}) - \cos(2\pi d(u+v) + \frac{2\pi}{3})) & -8\pi^2 d^2 \cos(2\pi d(u+v) + \frac{2\pi}{3}) \\ -8\pi^2 d^2 \cos(2\pi d(u+v) + \frac{2\pi}{3}) & -8\pi^2 d^2 (\cos(2\pi dv - \frac{2\pi}{3}) - \cos(2\pi d(u+v) + \frac{2\pi}{3})) \end{pmatrix}$$

has full rank in all the critical points, hence they are non-degenerate. By adding the critical points in (a), (b) and (c), we obtain that H_d has $(d-1)^2$ critical points in the interior of Δ . \Box

In Figs. 1, 2, we can see the critical points of H_d inside Δ for d = 6, 9. Critical points with critical values 6, -2, -3 are represented by \circ , *, •, respectively. The distance between two consecutive lines in the (u, v) oblique coordinate system is $\frac{1}{6d}$.



Fig. 1. The critical points of H_6 inside the fundamental region Δ of the affine Weyl group $\tilde{W}(A_2)$.



Fig. 2. The critical points of H_9 inside Δ .

Theorem. The number of singular points of U_d is:

$$\binom{d}{2} \left\lfloor \frac{d}{2} \right\rfloor + \left(\frac{d^2}{3} - d + 1 \right) \left\lfloor \frac{d - 1}{2} \right\rfloor$$
(8)

All the singular points are non-degenerate. The surface cannot have singular points at infinity.

Proof. The Chebyshev polynomials $T_d(w)$ have $\lfloor \frac{d}{2} \rfloor$ critical points with critical value 2 and $\lfloor \frac{d-1}{2} \rfloor$ critical points with critical value 3. The surface is singular at the points where the sum of the critical values of $T_d(w)$ and $Q_d(u, v)$ is zero. The result for the number of non-degenerate singularities of U_d follows then from the Lemma: Q_d has $\binom{d}{2}$ critical points with critical value -2 and $\frac{d^2}{3} - d + 1$ critical points with critical value -3. In the Lemma, we have also shown that the number of distinct critical points of Q_d is $(d-1)^2$, therefore U_d can not

have singular points at infinity. \Box

Consider a surface of degree d in $\mathbf{P}^3(\mathbf{C})$ with N(d) double points and no other singularities, and let $\mu(d) = \max N(d)$. Then we have:

Corollary.
$$\mu(3n) \ge {3n \choose 2} \lfloor \frac{3n}{2} \rfloor + (3n^2 - 3n + 1) \lfloor \frac{3n-1}{2} \rfloor.$$

We notice that U_{3n} has $\lfloor \frac{3n-1}{2} \rfloor$ more singularities than V_{3n} (see Eq. (3)). Also of interest are hypersurfaces in $\mathbf{P}^4(\mathbf{C})$ with affine equations:

$$Q_{3n}(u_1, u_2) - Q_{3n}(u_3, u_4) = 0 \tag{9}$$

They have $(\frac{3n(3n-1)}{2})^2 + (3n(n-1)+1)^2 + (\frac{3n(n-1)}{2})^2$ non-degenerate singularities. We find 3n(n-1) more singularities than in the Chmutov hypersurfaces $P_{3n}(u_1, u_2) - P_{3n}(u_3, u_4) = 0$. Hypersurfaces with A_j -singularities in $\mathbf{P}^n(\mathbf{C})$ can be obtained

along the lines of [5]. In particular, there is a family of Belyi polynomials associated with a series of planar trees obtained by a substitution process, which, when used in combination with $Q_d + 2$, allows us to show the existence of surfaces with a high number of cusps (see Appendix B and Fig. 1 in [5]).

Real variants of V_d were studied in [1], and the authors showed that the known lower bounds for the maximum number of ordinary double points on a surface of degree d can be attained with only real singularities. Recently, we have shown that a construction connected with the derivation of substitution tilings [3] can be used for the generation of algebraic surfaces with many real nodes [4]. One of the two types of surfaces obtained is equivalent to real variants of V_d . The other type consists in surfaces of degree 3n that have the same number of singularities as the surfaces presented in this work. In fact, they are related to the real variants of $U_{3n}(u, v, w)$ with u = x + iy, v = x - iy, w = z, for $x, y, z \in \mathbf{R}$.

We have considered only the polynomials Q_d for d = 3n. The study of the critical points of Q_d for $d \neq 3n$ shows that the Chmutov lower bound $\mu(d) \ge {d \choose 2} \lfloor \frac{d}{2} \rfloor + (\frac{1}{3}d^2 - d + \frac{2}{3}) \lfloor \frac{d-1}{2} \rfloor$ is not improved for such cases with surfaces of the type described by Eq. (1).

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