Algebraic geometry

# On a family of complex algebraic surfaces of degree $3 n$ 

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## Sur une famille de surfaces algébriques complexes de degré $3 n$

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#### Abstract

We study a class of algebraic surfaces of degree $3 n$ in the complex projective space with only ordinary double points. They are obtained by using bivariate polynomials with complex coefficients related to the generalized cosine associated with the affine Weyl group of the root system $A_{2}$.


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R É S U M É
Nous étudions une classe de surfaces algébriques de degré $3 n$ dans l'espace projectif complexe, avec seulement des points doubles ordinaires. Ils sont générés par des polynômes complexes qui sont liés au cosinus généralisé associé au groupe de Weyl affine du système de racines $A_{2}$.
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In [2], Chmutov introduced surfaces $V_{d}$ of degree $d$ in the complex projective space $\mathbf{P}^{3}(\mathbf{C})$ given by the affine equations $P_{d}(u, v)+T_{d}(w)=0$, where $P_{d} \mathrm{~s}$ are the generalized Chebyshev polynomials or folding polynomials [6,7] associated with the affine Weyl group $\tilde{W}\left(A_{2}\right)$ of the root system $A_{2}$, and $T_{d}$ is a Chebyshev polynomial. In this work, we study a class of surfaces of degree $3 n$ that have more singularities than the Chmutov surfaces of the same degree.

For $d=3 n, n=1,2,3, \ldots$ the surfaces $U_{d}$ are given by the affine equations:

$$
\begin{equation*}
Q_{d}(u, v)+T_{d}(w)=0 \tag{1}
\end{equation*}
$$

where $T_{d}$ is the Chebyshev polynomial of degree $d$ with two critical values 2 and 3 , and $Q_{d}$ is defined as follows. The generalized cosine associated with the affine Weil group $\tilde{W}\left(A_{2}\right)$ is $h(u, v)=\left(h_{1}, h_{2}\right)$, where:

$$
\begin{equation*}
h_{1}(u, v):=\mathrm{e}^{-2 \pi \mathrm{i} u}+\mathrm{e}^{-2 \pi \mathrm{i} v}+\mathrm{e}^{2 \pi \mathrm{i}(u+v)}, \quad h_{2}(u, v):=\mathrm{e}^{2 \pi \mathrm{i} u}+\mathrm{e}^{2 \pi \mathrm{i} v}+\mathrm{e}^{-2 \pi \mathrm{i}(u+v)} \tag{2}
\end{equation*}
$$

The polynomials $P_{d}$ appearing in the surfaces $V_{d}$ are defined in such a way that $P_{d}\left(h_{1}(u, v), h_{2}(u, v)\right):=C_{d}(u, v)$, where $C_{d}(u, v):=h_{1, d}(u, v)+h_{2, d}(u, v), h_{1, d}(u, v)=h_{1}(d u, d v), h_{2, d}(u, v)=h_{2}(d u, d v)$. For $(x, y):=h(u, v)$, it can be shown [6,2] that $P_{d}(x, y)=j_{d}(x, y)+j_{d}(y, x)$, where $j_{d}(x, y)$ is the determinant of a $d x d$ matrix:

[^0]\[

j_{d}(x, y)=\operatorname{det}\left($$
\begin{array}{cccccccc}
x & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\
2 y & x & 1 & 0 & 0 & 0 & \ldots & 0 \\
3 & y & x & 1 & 0 & 0 & \ldots & 0 \\
0 & 1 & y & x & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & y & x & 1 & 0 \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\ldots & \ldots & \ldots & 0 & 1 & y & x & 1 \\
0 & \ldots & \ldots & 0 & 0 & 1 & y & x
\end{array}
$$\right)
\]

The number of non-degenerate singularities of $V_{d}$ for $d=3 n$ is [2]:

$$
\begin{equation*}
\binom{d}{2}\left\lfloor\frac{d}{2}\right\rfloor+\left(\frac{d^{2}}{3}-d\right)\left\lfloor\frac{d-1}{2}\right\rfloor \tag{3}
\end{equation*}
$$

For the construction of the surfaces $U_{d}$, we use:

$$
\begin{equation*}
g_{1, d}(u, v):=\mathrm{e}^{-2 \pi \mathrm{i}\left(d u+\frac{2}{3}\right)}+\mathrm{e}^{-2 \pi \mathrm{i}\left(d v-\frac{1}{3}\right)}+\mathrm{e}^{2 \pi \mathrm{i}\left(d(u+v)+\frac{1}{3}\right)}, \quad g_{2, d}:=\mathrm{e}^{2 \pi \mathrm{i}\left(d u+\frac{2}{3}\right)}+\mathrm{e}^{2 \pi \mathrm{i}\left(d v-\frac{1}{3}\right)}+\mathrm{e}^{-2 \pi \mathrm{i}\left(d(u+v)+\frac{1}{3}\right)} \tag{4}
\end{equation*}
$$

There are bivariate polynomials $Q_{d}$ such that $Q_{d}\left(h_{1}(u, v), h_{2}(u, v)\right):=H_{d}(u, v)$, where:

$$
\begin{equation*}
H_{d}(u, v):=g_{1, d}(u, v)+g_{2, d}(u, v)=2 \cos \left(2 \pi d u-\frac{2 \pi}{3}\right)+2 \cos \left(2 \pi d v-\frac{2 \pi}{3}\right)+2 \cos \left(2 \pi d(u+v)+\frac{2 \pi}{3}\right) \tag{5}
\end{equation*}
$$

Having in mind that $g_{1, d}(u, v)=\mathrm{e}^{\frac{2 \pi i}{3}} h_{1, d}(u, v)$ and $g_{2, d}(u, v)=\mathrm{e}^{-\frac{2 \pi i}{3}} h_{2, d}(u, v)$, we get, for $(x, y):=h(u, v)$ :

$$
\begin{equation*}
Q_{d}(x, y)=\mathrm{e}^{\frac{2 \pi i}{3}} \mathrm{j}_{d}(x, y)+\mathrm{e}^{-\frac{2 \pi i}{3}} \mathbf{j}_{d}(y, x) \tag{6}
\end{equation*}
$$

The homogeneous part of highest degree in $Q_{d}(x, y)$ is $\mathrm{e}^{\frac{2 \pi i}{3}} x^{d}+\mathrm{e}^{-\frac{2 \pi i}{3}} y^{d}$; therefore, $Q_{d}$ has $(d-1)^{2}$ critical points.
Lemma. The polynomial $Q_{d}$ has $\frac{d(d-3)}{6}$ critical points with critical value $6,\binom{d}{2}$ critical points with critical value -2 and $\frac{d^{2}}{3}-d+1$ critical points with critical value -3 . All the critical points of $Q_{d}$ are non-degenerate.

Proof. We restrict $H_{d}$ and $h$ on the plane with real coordinates. The Jacobi matrices satisfy $J\left(H_{d}\right)=J\left(Q_{d}\right) \circ J(h)$. Zeros of the Jacobian determinant of $h$ :

$$
\begin{equation*}
\operatorname{det} J(h)=4 \pi^{2} \mathrm{e}^{-2 \pi \mathrm{i}(u+v)}\left(\mathrm{e}^{-2 \pi \mathrm{i} v}-\mathrm{e}^{-2 \pi \mathrm{i} u}\right)\left(\mathrm{e}^{2 \pi \mathrm{i}(u+2 v)}-1\right)\left(\mathrm{e}^{2 \pi \mathrm{i}(2 u+v)}-1\right) \tag{7}
\end{equation*}
$$

are the sides of the triangle $\Delta$ whose interior is given by $u-v>0, u+2 v>0,2 u+v<1$, which is the fundamental region of $\tilde{W}\left(A_{2}\right)$. All the points from one orbit of $\tilde{W}\left(A_{2}\right)$ are mapped by $h$ into a single point, and the images of the critical points of $H_{d}$ from the interior of $\Delta$ by $h$ are the critical points of $Q_{d}$.

A direct computation of the critical points of $H_{d}$ leads to three cases. In the following list, we indicate the critical value $\zeta$ and the number of points $N_{\zeta}$ corresponding to $\zeta$ inside $\Delta$ by $\left(\zeta, N_{\zeta}\right)$ :
(a) $\left(6, \frac{d(d-3)}{6}\right) ; u=\frac{3 k+1}{d}, v=\frac{3 l+1}{d} ; k, l \in \mathbf{Z}$.
(b) $\left(-3, \frac{d^{2}}{3}-d+1\right) ; u=\frac{k}{3 d}, v=\frac{l}{3 d}$, with $k=3 m-1, l=3 p-1$ or $k=3 m, l=3 p ; m, p \in \mathbf{Z}$.
(b1) $\left(-3,1+\frac{d(d-3)}{6}\right) ; u=\frac{3 m+2}{3 d}, v=\frac{3 p+2}{3 d} ; m, p \in \mathbf{Z}$.
(b2) $\left(-3, \frac{d(d-3)}{6}\right) ; u=\frac{m}{d}, v=\frac{p}{d} ; m, p \in \mathbf{Z}$.
(c) $\left(-2,\binom{d}{2}\right) ; u=\frac{3 k+2}{6 d}, v=\frac{3 l+2}{6 d}$ with $k$ or $l$ odd.
(c1) $\left(-2, \frac{d(d-1)}{3}\right) ; u=\frac{6 m-1}{6 d}, v=\frac{3 p-1}{6 d} ; m, p \in \mathbf{Z}$.
(c2) $\left(-2, \frac{d(d-1)}{6}\right) ; u=\frac{6 m+2}{6 d}, v=\frac{6 p-1}{6 d} ; m, p \in \mathbf{Z}$.
The hessian matrix of $H_{d}$ :

$$
\left(\begin{array}{cc}
-8 \pi^{2} d^{2}\left(\cos \left(2 \pi d u-\frac{2 \pi}{3}\right)-\cos \left(2 \pi d(u+v)+\frac{2 \pi}{3}\right)\right) & -8 \pi^{2} d^{2} \cos \left(2 \pi d(u+v)+\frac{2 \pi}{3}\right) \\
-8 \pi^{2} d^{2} \cos \left(2 \pi d(u+v)+\frac{2 \pi}{3}\right) & -8 \pi^{2} d^{2}\left(\cos \left(2 \pi d v-\frac{2 \pi}{3}\right)-\cos \left(2 \pi d(u+v)+\frac{2 \pi}{3}\right)\right)
\end{array}\right)
$$

has full rank in all the critical points, hence they are non-degenerate. By adding the critical points in (a), (b) and (c), we obtain that $H_{d}$ has $(d-1)^{2}$ critical points in the interior of $\Delta$.

In Figs. 1, 2, we can see the critical points of $H_{d}$ inside $\Delta$ for $d=6,9$. Critical points with critical values $6,-2,-3$ are represented by $\circ, *$, $\bullet$, respectively. The distance between two consecutive lines in the $(u, v)$ oblique coordinate system is $\frac{1}{6 d}$.


Fig. 1. The critical points of $H_{6}$ inside the fundamental region $\Delta$ of the affine Weyl group $\tilde{W}\left(A_{2}\right)$.


Fig. 2. The critical points of $H_{9}$ inside $\Delta$.
Theorem. The number of singular points of $U_{d}$ is:

$$
\begin{equation*}
\binom{d}{2}\left\lfloor\frac{d}{2}\right\rfloor+\left(\frac{d^{2}}{3}-d+1\right)\left\lfloor\frac{d-1}{2}\right\rfloor \tag{8}
\end{equation*}
$$

All the singular points are non-degenerate. The surface cannot have singular points at infinity.
Proof. The Chebyshev polynomials $T_{d}(w)$ have $\left\lfloor\frac{d}{2}\right\rfloor$ critical points with critical value 2 and $\left\lfloor\frac{d-1}{2}\right\rfloor$ critical points with critical value 3. The surface is singular at the points where the sum of the critical values of $T_{d}(w)$ and $Q_{d}(u, v)$ is zero. The result for the number of non-degenerate singularities of $U_{d}$ follows then from the Lemma: $Q_{d}$ has $\binom{d}{2}$ critical points with critical value -2 and $\frac{d^{2}}{3}-d+1$ critical points with critical value -3 .

In the Lemma, we have also shown that the number of distinct critical points of $Q_{d}$ is $(d-1)^{2}$, therefore $U_{d}$ can not have singular points at infinity.

Consider a surface of degree $d$ in $\mathbf{P}^{3}(\mathbf{C})$ with $N(d)$ double points and no other singularities, and let $\mu(d)=\max N(d)$. Then we have:

Corollary. $\mu(3 n) \geqslant\binom{ 3 n}{2}\left\lfloor\frac{3 n}{2}\right\rfloor+\left(3 n^{2}-3 n+1\right)\left\lfloor\frac{3 n-1}{2}\right\rfloor$.
We notice that $U_{3 n}$ has $\left\lfloor\frac{3 n-1}{2}\right\rfloor$ more singularities than $V_{3 n}$ (see Eq. (3)). Also of interest are hypersurfaces in $\mathbf{P}^{4}(\mathbf{C})$ with affine equations:

$$
\begin{equation*}
Q_{3 n}\left(u_{1}, u_{2}\right)-Q_{3 n}\left(u_{3}, u_{4}\right)=0 \tag{9}
\end{equation*}
$$

They have $\left(\frac{3 n(3 n-1)}{2}\right)^{2}+(3 n(n-1)+1)^{2}+\left(\frac{3 n(n-1)}{2}\right)^{2}$ non-degenerate singularities. We find $3 n(n-1)$ more singularities than in the Chmutov hypersurfaces $P_{3 n}\left(u_{1}, u_{2}\right)-P_{3 n}\left(u_{3}, u_{4}\right)=0$. Hypersurfaces with $A_{j}$-singularities in $\mathbf{P}^{n}(\mathbf{C})$ can be obtained
along the lines of [5]. In particular, there is a family of Belyi polynomials associated with a series of planar trees obtained by a substitution process, which, when used in combination with $Q_{d}+2$, allows us to show the existence of surfaces with a high number of cusps (see Appendix B and Fig. 1 in [5]).

Real variants of $V_{d}$ were studied in [1], and the authors showed that the known lower bounds for the maximum number of ordinary double points on a surface of degree $d$ can be attained with only real singularities. Recently, we have shown that a construction connected with the derivation of substitution tilings [3] can be used for the generation of algebraic surfaces with many real nodes [4]. One of the two types of surfaces obtained is equivalent to real variants of $V_{d}$. The other type consists in surfaces of degree $3 n$ that have the same number of singularities as the surfaces presented in this work. In fact, they are related to the real variants of $U_{3 n}(u, v, w)$ with $u=x+i y, v=x-i y, w=z$, for $x, y, z \in \mathbf{R}$.

We have considered only the polynomials $Q_{d}$ for $d=3 n$. The study of the critical points of $Q_{d}$ for $d \neq 3 n$ shows that the Chmutov lower bound $\mu(d) \geqslant\binom{ d}{2}\left\lfloor\frac{d}{2}\right\rfloor+\left(\frac{1}{3} d^{2}-d+\frac{2}{3}\right)\left\lfloor\frac{d-1}{2}\right\rfloor$ is not improved for such cases with surfaces of the type described by Eq. (1).

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