Automation (theoretical)

Growth bound of delay-differential algebraic equations

Taux de croissance des équations algébro-différentielles à retards

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1. Introduction

Delay-differential algebraic equations (DDAE) are a class of functional differential equations (FDE) whose variables are connected through integrators and finite-memory delay operators. Such a system of equations, with variables \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \) and a finite memory length \( r \), has the structure:

\[
\begin{align*}
\dot{x}(t) &= Ax_t + By_t \\
y(t) &= Cx_t + Dy_t
\end{align*}
\]

(1)

where \( z_t \) refers to the memory of the variable \( z \) at time \( t \):

\[
\text{dom} z_t = [-r, 0] \quad \text{and} \quad \forall \theta \in [-r, 0], \ z_t(\theta) = z(t+\theta)
\]

(2)

and the symbols \( A, B, C, D \) denote delay operators:

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} : C([-r, 0], \mathbb{C}^{n+m}) \to \mathbb{C}^{n+m}, \quad \text{linear and bounded.}
\]

(3)
This class of systems has been already considered in [1,2,4,10–12], but most of the literature has been focused on some of its subclasses: equations of retarded type, neutral type and difference equations. Nonetheless, the general model is important: it is required in the modelling of some physical phenomena such as lossless propagation (see [11] and the references therein) and in the control of dead-time systems when standard methods such as finite-spectrum assignment [9] are used.

A classic stability criterion determines the growth bound of a DDAE system from the location of the rightmost zeros of its characteristic function. The validity of this criterion has already been established with several methods under various restrictive assumptions: for systems of retarded type, for systems of neutral type, and difference equations whose difference operator combines only discrete and distributed delays [7] or satisfies a “jump” assumption [6], and for DDAE with discrete delays and stable difference operators [5].

We demonstrate in this paper that a single method, that combines the use of the Gearhart–Prüss theorem with bounds for the characteristic matrix inverse, established by complex analysis, can be used to prove this criterion in all these special cases. Actually, we require only the DDAE system to have a strictly causal difference operator, an assumption already used to ensure the well-posedness of the system. To the best knowledge of the author, this general result was not available.

Matrix-valued measures provide a concrete representation of delay operators: any linear bounded operator $L : C([-r,0],C^l) → C^l$ corresponds to a unique countably additive function on the bounded Borel subsets of $R$, supported on $[-r,0]$, with values in $C^{l×l}$. This alternate representation – that we still denote by $L$ – is related to the initial operator by:

$$L\phi = ∫ dL\phi := ∑_l [∫ \phi_k dL_{lk}] e_l$$

(4)

where $(e_1, ..., e_l)$ denotes the canonical basis of $C^l$. Let $L^*$ be the measure obtained by symmetry around $t = 0$ of $L$, such that for any bounded Borel set $B$, $L^*(B) = L((-B)$ and let $*$ be the convolution between time-dependent locally integrable functions – or more generally measures – of left-sided bounded support. The convolution of two scalar, vector or matrix-valued measures of compatible dimensions is defined as the combination of scalar convolution and linear algebra product; for example, for two matrix-valued measures $A$ and $B$, $A * B$ is the matrix-valued measure such that $(A * B)_{ij} := ∑_k A_{ik} * B_{kj}$.

We also implicitly extend functions defined on a subset of $R$ by zero outside of their domain. With these conventions, for any continuous function $z$ defined on $[-r, +∞)$, we have $∀ t > 0, Lz = (L^* z)(t)$. As the right-hand side of this equation is still properly defined – as a locally integrable function of $t$ – if $z$ is merely locally integrable, we may rewrite Eq. (1) as a convolution equation.

Let $e$ be the Heaviside function. We say that a pair of locally integrable functions $(x, y)$, defined on $[-r, +∞)$, with values in $C^{l+m}$, is a (locally integrable) solution of (1) if there is an $f ∈ C^m$ such that:

$$\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} e * A^* e * B^* \\ C^* D^* \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}(t) + \begin{bmatrix} f \\ 0 \end{bmatrix} \text{ for a.e. } t > 0.$$

(5)

We assume in the sequel that the difference operator $D$ is strictly causal, that is $D((0]) = 0$. This condition ensures that this system of equations defines a well-posed initial value problem in the Hilbert product space $X = C^m × L^2([-r,0],C^{l+m})$, see [1,2,12]: given any $(\phi, χ, ψ) ∈ X$, there is a unique solution $(x,y)$ such that $(x(0^+), x_0, y_0) = (\phi, χ, ψ)$ and the mapping $(t ∈ R_+ → \exp(At))$ given by $(t(0^+), x(t), y(t)) = \exp(At)(\phi, χ, ψ)$ for $t ≥ 0$ is a strongly continuous semigroup on $X$.

2. Exponential solutions – characteristic matrix and resolvent operator

We denote by $Δ$ the characteristic matrix of system (1), defined at any point $s ∈ C$ by

$$Δ(s) = \begin{bmatrix} sI_m & 0 \\ 0 & I_m \end{bmatrix} - \mathcal{L} \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}(s)$$

(6)

where $\mathcal{L}$ is the Laplace transform.

The determinant of the characteristic matrix – the characteristic function – and its adjugate both have a quasi-polynomial structure:

$$d \Delta(s) = ∑_{i=0}^n c_i(s)s^i, \quad \text{adj } Δ(s) = ∑_{i=0}^n C_i(s)s^i$$

(7)

where the $c_i$ (resp. $C_i$) are entire functions (resp. matrices of entire functions) bounded on any right-hand plane. Moreover, the leading coefficient of the characteristic function is given by $c_n(s) = \det Δ_0(s)$, where $Δ_0$ is the characteristic matrix of the system $y(t) = Dy_t$, Lemma 2.1 establishes elementary properties of $\det Δ_0$ and Lemma 2.2 describes how the zeros of $\det Δ_0$ and $\det Δ$ are connected.

For any real number $σ$, we denote by $P_σ$ the open half-plane $\{ s ∈ C \mid \Re(s) > σ \}$ and for any positive $η$, we denote by $Z_η$ the set of complex numbers whose distance to the zeros of $\det Δ_0$ it at most $η$:

$$Z_η = \{ s ∈ C \mid ∃ z ∈ C, \det Δ_0(z) = 0 ∧ \text{dist}(z, S) ≤ η \}.$$ 

(8)
Lemma 2.1 [Zero clusters and lower bound]. Let $\sigma \in \mathbb{R}$ and $\epsilon > 0$. There is an $\eta > 0$ such that any connected component $\Lambda$ of the set $Z_\eta$ is bounded and $\Lambda \subset P_{\sigma - \epsilon}$ if $\Lambda \cap P_{\sigma} \neq \emptyset$. Moreover, for any $\eta > 0$, there is a $\kappa > 0$ such that $|\det \Delta_0| \geq \kappa$ on $P_{\sigma - \epsilon} - Z_\eta$.

**Proof.** Let $\Sigma_m$ be the set of permutations of $\{1, \ldots, m\}$ and

$$\det_s M = \sum_{\sigma \in \Sigma_m} \text{sgn}(\sigma) M_{1, \sigma(1)} \cdots M_{m, \sigma(m)}.$$  

As $\Delta_0(s) = I_m - L^*D^*(s)$, $\det \Delta_0 = L\mu$ where $\mu = \det_s(\delta_0 I_m - D^*)$. The complex measure $\mu$ is a sum of convolution products of $m$ complex measures supported on $[0, r]$, hence it is supported on $[0, mr]$. Consequently, $\det \Delta_0$ is an entire function that satisfies the inequality:

$$|\det \Delta_0(s)| \leq |\mu([0, mr])| \max(1, \exp(-\Re(s)mr)).$$  

(9)

Since $D([0]) = 0$, we also have $\mu([0]) = 1$, which yields:

$$\lim_{\eta \to +\infty} \det \Delta_0(s) = 1.$$

(10)

The function $z \mapsto \det \Delta_0(iz)$ meets the assumptions of [8, th. VIII]. Thus, the number of distinct zeros $N(\rho)$ of $\det \Delta_0$ whose modulus is less than $\rho$ is such that $\limsup_{\rho \to +\infty} N(\rho)/\rho \leq 2mr/\pi$. If there is an unbounded connected component of $Z_\eta$, there is a corresponding sequence $(zn)_{n \in \mathbb{N}}$ of distinct zeros of $\det \Delta_0$ such that, for any $n \in \mathbb{N}$, $|zn + 1 - zn| \leq 2\eta$, which provides the density estimate $\limsup_{\rho \to +\infty} N(\rho)/\rho \geq 1/2n$. Consequently, if $\eta < \pi/4mr$, every connected component of $Z_\eta$ is bounded.

The proof of the two remaining statements uses the same complex analysis argument. Consider a sequence $s_n$ of numbers in $P_{\sigma - \epsilon}$ such that $\eta s_n \to x \in \mathbb{R}$ when $n \to +\infty$. From (9), it follows that the sequence of functions defined by $f_n(s) = \det \Delta_0(s + i\eta s_n)$ is locally uniformly bounded on $C$. By Montel’s theorem, a subsequence converges locally uniformly to an entire function $f_\infty$, which by (10) is not identically zero. From Hurwitz’s theorem, it follows that for any sufficiently small $\alpha > 0$, there is an arbitrarily large integer $n$ such that $\det \Delta_0$ has $m$ zeros in the open disk $B(s_n, \alpha)$, where $m$ is the multiplicity of $x$ if $f_\infty(x) = 0$, or 0 otherwise.

Consider a sequence $\Lambda_n$ of bounded connected components of $Z_{1/2n}$, defined for $n$ sufficiently large, such that $\Lambda_n \cap P_{\sigma} \neq \emptyset$ and $\Lambda_n - P_{\sigma - \epsilon} \neq \emptyset$. For any such $n$ and any $\alpha < \epsilon$, there is a $y_n \in \mathbb{R}$ such that the number of zeros of $\det \Delta_0$ in the open disk $B(0, \alpha)$ is greater than $\eta y_n - 1$. This contradicts the result of the previous paragraph for the sequence $s_n = \sigma + iy_n$.

Finally, if $|\det \Delta_0|$ has no positive lower bound on $P_{\sigma - \epsilon} - Z_\eta$, we can find in this set a sequence $s_n$ such that $\det \Delta_0(s_n) \to 0$ when $n \to +\infty$. By (10), the sequence $s_n$ can be selected such that $\eta s_n$ has a limit $x \in \mathbb{R}$. As $f_\infty(x) = \lim_{n \to +\infty} f_n(s_n - i\eta s_n) = 0$, there is an integer $n$ such that $\det \Delta_0$ has at least one zero in $B(s_n, \eta)$, a contradiction with the assumption that $s_n \notin Z_\eta$. \square

Lemma 2.2 [Characteristic function zeros]. Let $\sigma \in \mathbb{R}$. If the function $\det \Delta_0$ has an infinite number of zeros on $P_{\sigma}$, the function $\det \Delta$ has an infinite number of zeros on $P_{\sigma - \epsilon}$ for any $\epsilon > 0$.

**Proof.** Suppose that $\det \Delta_0$ has an infinite number of zeros on $P_{\sigma}$. Let $\eta > 0$ be such that any connected component $\Lambda$ of $Z_\eta$ that contains such a zero is bounded and is included in $P_{\sigma - \epsilon}$ (Lemma 2.1). The zeros of $\det \Delta_0$ are isolated, hence every $\Lambda$ contains a finite number of zeros, and the collection of sets $\Lambda$ is therefore infinite. It is also locally finite: for any compact set $K \subset \mathbb{C}$ and any set $\Lambda$ such that $K \cap \Lambda \neq \emptyset$, $\Lambda$ contains a closed disk $B_\eta$ of radius $\eta$ such that $K \cap B_\eta \neq \emptyset$, hence $B_\eta \subset K' = K + \overline{B}(0, \eta)$ and only a finite number of such disjoint disks $B_\eta$ may be contained in $K'$. Thus, for any $\rho > 0$, there is a set $\Lambda$ that does not intersect $\overline{B}(0, \rho)$.

The lower bound from Lemma 2.1 and the quasi-polynomial structure of $\det \Delta_0$ provide the existence of a $\rho_0 > 0$ such that, if $s \in P_{\sigma - \epsilon}$ is not in $Z_{\eta/2}$ and satisfies $|s| > \rho_0$, then $|\det \Delta(s) - s^\alpha \det \Delta_0(s)| < |s^\alpha \det \Delta_0(s)|$. Let $\Lambda_0$ be one of the sets $\Lambda$ that does not intersect $\overline{B}(0, \rho_0)$. As $\Lambda_0$ is included in $P_{\sigma - \epsilon}$, the application of Rouché’s theorem to its boundary yields the existence of at least one zero of $\det \Delta$ in $P_{\sigma - \epsilon}$. We may more generally define $\rho_{n+1} = \sup\{|s|, s \in \Lambda_n\}$ and apply the same argument to a set $\Lambda_{n+1}$ that does not intersect $\overline{B}(0, \rho_{n+1})$ to prove the existence of an infinite number of zeros of $\det \Delta$ in $P_{\sigma - \epsilon}$. \square

The infinitesimal generator $\mathcal{A}$ of the DDAE semigroup is defined by $\mathcal{A}(\phi, \chi, \psi) = (A\chi + B\psi, \dot{\chi}, \dot{\psi})$ on the domain \{$(\phi, \chi, \psi) \in C^0 \times W^{1,2}([-r, 0], C^{m+m}) | \chi(0) = \phi, \psi(0) = C\chi + D\psi$\}, see [12]. The resolvent operator $(sI - \mathcal{A})^{-1}$ exists iff $\Delta(s)^{-1}$ exists: the resolvent set of $\mathcal{A}$ is:

$$\rho(\mathcal{A}) = \{s \in \mathbb{C}, \ker \Delta(s) = \{0\}\}.$$

(11)

Moreover, for any real number $\sigma$, there are constants $\kappa_\sigma$ and $\lambda_\sigma$ such that:

$$\|sI - \mathcal{A}\| \leq \kappa_\sigma \|\Delta(s)^{-1}\| + \lambda_\sigma \quad \text{if} \forall s \geq \sigma \text{ and } s \in \rho(\mathcal{A}).$$

(12)
Let the growth bound $\omega_0$ of the DDAE system be the infimum of the real numbers $\omega$ for which there exists $\alpha > 0$ such that, for any initial condition $(x(0^+), x_0, y_0) \in X$, the solution of the system satisfies:

$$\forall t \geq 0, \quad \|x(t^+), x_t, y_t\|_X \leq \alpha \exp(\omega t) \|x(0^+), x_0, y_0\|_X. \quad (13)$$

**Theorem 2.3** (Growth bound). The growth bound of a DDAE system such that $D(0) = 0$ is determined by the rightmost zeros of the characteristic function and given by:

$$\omega_0 = \sup \{ \Re(s), \quad s \in \mathbb{C} \mid \det \Delta(s) = 0 \}. \quad (14)$$

As a corollary, such a DDAE system is uniformly exponentially stable iff the set of zeros of its characteristic function is on the left of – and at a positive distance from – the imaginary axis.

**Proof.** We use the Gearhart–Prüss theorem [3] to establish the result: we prove that for any $\sigma > s(A)$, where $s(A)$ is the spectral bound of $A$, $\|(sI - A)^{-1}\|$ is bounded on $P_\sigma$; this is achieved by combining inequality (12) with the derivation of a bound for $\|\Delta(s)^{-1}\|$ on $P_\sigma$.

The quasi-polynomial structure of the adjugate matrix yields on $P_\sigma$ the estimate $\|\text{adj} \Delta(s)\| \leq \kappa (1 + |s|^n)$. From (11) we deduce that $\det \Delta$ has no zero on $P_{\sigma-\epsilon}$ for any $\epsilon > 0$ such that $s(A) < \sigma - \epsilon$ and hence, by Lemma 2.2, $\det \Delta_0$ has at most a finite number of zeros on $P_{\sigma-\epsilon}$ and therefore on $P_\sigma$. On $P_\sigma$ and away from these zeros, $|\det \Delta_0|$ has a positive lower bound $\kappa'$ by Lemma 2.1. It follows from the quasi-polynomial structure of $\det \Delta$ that $|\det \Delta(s)| \geq \kappa'' (1 + |s|^n)$ for a $\kappa'' > 0$, on $P_\sigma$, except on a compact set $K$ and by continuity, this estimate still holds on all of $P_\sigma$ with a possibly smaller $\kappa''$. Finally, for any $s \in P_\sigma$, $\|\Delta(s)^{-1}\| = \|\text{adj} \Delta(s)\|/|\det \Delta(s)| \leq \kappa / \kappa''$. \quad \square

**References**


