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Partial Differential Equations

Global existence and boundedness of classical solutions for a chemotaxis model with logistic source

Existence globale et bornes des solutions classiques d'un modèle chimiotaxique avec une source logistique

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ABSTRACT

We consider the chemotaxis system:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi(v)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + ug(u), & x \in \Omega,\ t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, with smooth boundary and function f is assumed to generalize the logistic source:

$$f(u) = au - bu^2, \quad u \geqslant 0, \text{ with } a > 0,\ b > 0.$$

Moreover, $\chi(s)$ and $g(s)$ are nonnegative smooth functions and satisfy:

$$\chi(s) \leqslant \frac{\varrho}{(1+\vartheta s)^k}, \quad s \geqslant 0, \text{ with some } \varrho > 0,\ \vartheta > 0 \text{ and } k > 1,$$

$$g(s) \leqslant \frac{h_0}{(1+hs)^\delta}, \quad s \geqslant 0, \text{ with } h_0 > 0,\ h \geqslant 0,\ \delta \geqslant 0.$$

We prove that for all positive values of ϱ, a and b, classical solutions to the above system are uniformly-in-time bounded. This result extends a recent result by C. Mu, L. Wang, P. Zheng and Q. Zhang (2013) [13], which asserts the global existence and boundedness of classical solutions on condition that $0 \leqslant a < 2b$ and ϱ be sufficiently small.

RÉSUMÉ

On considère le système de la chimiotaxie :

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi(v)\nabla v) + f(u), & x \in \Omega, t > 0, \\ v_t = \Delta v - v - ug(u), & x \in \Omega, t > 0, \end{cases}$$

avec conditions de Neumann homogènes dans un domaine borné $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, de frontière régulière ; on suppose que f est une généralisation d'une source logistique :

$$f(u) = au - bu^2, \quad u \geqslant 0, \text{ avec } a > 0, b > 0.$$

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De plus, $\chi(s)$ et $g(s)$ sont des fonctions positives ou nulles régulières vérifiant :

$$\chi(s) \leqslant \frac{\varrho}{(1+\vartheta s)^k}, \quad s \geqslant 0, \text{ avec } \varrho > 0, \vartheta > 0, k > 1,$$

$$g(s) \leqslant \frac{h_0}{(1+hs)^\delta}, \quad s \geqslant 0, \text{ avec } h_0 > 0, \ h \geqslant 0, \delta \geqslant 0.$$

On démontre que, pour toute valeur positive de ϱ, a et b, les solutions classiques du système ci-dessus sont uniformément bornées en temps. Ce résultat étend un résultat récent de C. Mu, L. Wang, P. Zheng et Q. Zhang (2013) [13], qui établit l'existence globale et des bornes des solutions classiques sous les conditions $0 \leqslant a < 2b$ et ϱ assez petit.

1. Introduction

In this paper, we investigate the global existence and boundedness of solutions to the parabolic system:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi(v)\nabla v) + f(u), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + ug(u), & x \in \Omega, \ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0, \quad v(x,0) = v_0, & x \in \Omega, \end{cases} \tag{1.1}$$

in a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 1$, where ν denotes the unit outward normal vector to $\partial\Omega$ and u_0 and v_0 are given nonnegative functions. Moreover, the functions $\chi(s)$, $f(s)$ and $g(s)$ are smooth and $\chi(s)$ and $g(s)$ are nonnegative for $s \geqslant 0$.

In mathematical biology, systems like (1.1) are used to describe the movement of cells towards higher concentration of a chemical signal substance produced by the cells themselves. Such a movement is called chemotaxis. Here, $u = u(x,t)$ denotes the cell density and $v = v(x,t)$ is the concentration of the chemical substance. While the function $\chi(v)$ is the chemotactic sensitivity function, $f(u)$ is the growth of u and $ug(u)$ is the production of v [5,11,12].

Several types of functions have been proposed for $\chi(v)$, $f(u)$ and $g(u)$ (see [5,17,18,25]). For $f = 0$ and χ and g constants: it is known that when $n = 1$, the blow-up never can occur [19]. For $n = 2$, it has been shown in [16] that if $\|u_0\|_{L^1(\Omega)} < \frac{4\pi}{\chi g}$, then all solutions of (1.1) are global and bounded. The same result holds for radial solutions when $\|u_0\|_{L^1(\Omega)} < \frac{8\pi}{\chi g}$ [16]. While for $\|u_0\|_{L^1(\Omega)} > \frac{4\pi}{\chi g}$, solutions blow up either in finite or infinite time [8], moreover, the authors in [3] constructed radial solutions to (1.1) which blow up in finite time provided that $\|u_0\|_{L^1(\Omega)} > \frac{8\pi}{\chi g}$. For $n \geqslant 3$, Winkler [27] proved that, for each $q > \frac{n}{2}$ and $p > n$, there exists $\epsilon_0 > 0$ such that, if $\|u_0\|_{L^q(\Omega)} < \epsilon$ and $\|\nabla v_0\|_{L^p(\Omega)} < \epsilon$ for $\epsilon < \epsilon_0$, then solutions of (1.1) are global and bounded. Also, when Ω is a ball in $\mathbb{R}^n$, $n \geqslant 3$, the same author in [29] showed that, under some conditions on initial data and provided that $\|u_0\|_{L^1(\Omega)} > 0$, there exist radial solutions that blow up in finite time.

In [18], the authors proved the global existence of solutions of (1.1) for $n = 2$, g =constant, χ a smooth bounded function and:

$$f(u) = au - bu^2, \quad u \geqslant 0, \text{ with } -\infty < a < \infty, \ b > 0 \text{ and } \ f(0) = 0.$$

The authors in [17] proved the global existence of solutions of (1.1) in the two-dimensional case under the assumptions that χ is a positive constant and the functions $f(u)$ and $g(u)$ are smooth, and:

$$f(u) \leqslant 1 - bu^\alpha, \quad u \geqslant 0, \ b > 0, \ \alpha \geqslant 1,$$

$$0 \leqslant h(u) \leqslant (u+1)^\beta, \qquad 0 \leqslant h'(u) \leqslant \beta(u+1)^{\beta-1}, \quad u \geqslant 0, \ \beta > 0,$$

where $h(u) = ug(u)$. These results depend on the values of α and β. For boundedness of solutions, they showed that:

$$\|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{H^2(\Omega)} \leqslant \psi\big(\|u_0\|_{L^2(\Omega)} + \|v_0\|_{H^2(\Omega)}\big), \quad t \geqslant 0,$$

with some increasing function ψ.

For $n \geqslant 3$, Winkler in [25] proved that if $\chi \in \mathbb{R}$ constant, $g = 1$, Ω a bounded convex domain of $\mathbb{R}^n$ with smooth boundary and:

$$f(u) \leqslant a - bu^2, \quad u \geqslant 0, \text{ with } a > 0, \ b > 0 \text{ and } \ f(0) \geqslant 0,$$

then problem (1.1) admits a unique global classical solution, which is bounded in $\Omega \times (0,\infty)$ provided that b is sufficiently large.

There are much more works related to chemotaxis system in the literature; we refer the interested readers to [1,2,4,6,7,9,10,14,15,20–24,28,30] and the references therein.

Throughout this paper we assume that $\chi(v)$, $f(u)$ and $g(u)$ satisfy the following conditions:

$$\chi(v) \leqslant \frac{\varrho}{(1+\vartheta v)^k}, \quad v \geqslant 0, \text{ and some } \varrho > 0,\ \vartheta > 0 \text{ and } k > 1, \tag{1.2}$$

$$f(u) \leqslant au - bu^2, \quad u \geqslant 0, \text{ with } a > 0,\ b > 0 \text{ and } f(0) = 0, \tag{1.3}$$

$$g(u) \leqslant \frac{h_0}{(1+hu)^\delta}, \quad u \geqslant 0, \text{ with } h_0 > 0,\ h \geqslant 0,\ \delta \geqslant 0. \tag{1.4}$$

Winkler in [26] proved that if $f = 0$, $g = 1$ and χ satisfies the condition (1.2), then problem (1.1) has a unique global classical solution, which is uniformly bounded.

In [13], the authors showed that the solutions of problem (1.1) with (1.2), (1.3) and (1.4) are global and bounded provided that ϱ and a are sufficiently small. They left open the question of whether there exists a blow-up solution to problem (1.1), when the parameter a is sufficiently large. We answer this question and prove that solutions of problem (1.1) are global and bounded for all positive values of ϱ, a and b. This means that the blow-up never can occur for all positive values of ϱ, a and b.

2. Global existence

Let us state the standard well-posedness and classical solvability result.

Lemma 2.1. *Let the nonnegative functions u_0 and v_0 satisfy $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ for some $q > n$. Then problem (1.1) has a unique local-in-time nonnegative classical solution:*

$$u \in C^0\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big),$$
$$v \in C^0\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big) \cap L^\infty_{loc}\big([0, T_{\max}); W^{1,q}(\Omega)\big),$$

where $T_{\max}$ denotes the maximal existence time. In addition, if $T_{\max} < +\infty$, then:

$$\big\|u(.,t)\big\|_{L^\infty(\Omega)} + \big\|v(.,t)\big\|_{W^{1,q}(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}.$$

The proof follows from the general theory of parabolic systems. For details of the proof, we refer the reader to [9,26].

Lemma 2.2. *Suppose that f satisfies (1.3) with some $a, b > 0$. Then we have:*

$$\big\|u(.,t)\big\|_{L^1(\Omega)} \leqslant \frac{\max\{|\Omega|, \|u_0\|_{L^1(\Omega)}\}}{\min\{1, \frac{b}{a}\}}. \tag{2.1}$$

Proof. We integrate the first equation in (1.1) over Ω and using (1.3) to see that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)\,\mathrm{d}x = \int_\Omega f(u)\,\mathrm{d}x \leqslant a\int_\Omega u\,\mathrm{d}x - b\int_\Omega u^2\,\mathrm{d}x, \quad \text{for } t > 0.$$

From the Hölder inequality, we obtain $(\int_\Omega u\,\mathrm{d}x)^2 \leqslant |\Omega|\int_\Omega u^2\,\mathrm{d}x$, and hence $y(t) = \int_\Omega u(x,t)\,\mathrm{d}x$ satisfies:

$$y'(t) \leqslant ay(t) - b|\Omega|^{-1}y^2(t), \quad \text{for } t > 0.$$

Now, we set $\gamma(t) := y(t)^{-1}$, thus we obtain:

$$\gamma'(t) + a\gamma(t) \geqslant b|\Omega|^{-1},$$

which yields:

$$\gamma(t) \geqslant \mathrm{e}^{-at}\gamma(0) + \frac{b}{a}|\Omega|^{-1}\big(1 - \mathrm{e}^{-at}\big).$$

Therefore,

$$y(t) \leqslant \left(y(0)^{-1}\mathrm{e}^{-at} + \frac{b}{a}|\Omega|^{-1}\big(1 - \mathrm{e}^{-at}\big)\right)^{-1}.$$

This inequality yields:

$$y(t) \leqslant \frac{\max\{|\Omega|, y(0)\}}{\min\{1, \frac{b}{a}\}}.$$

So, the proof is complete. □

The main step towards the global existence of solutions is to establish a uniform bound of $u(.,t)$ in the space $L^{2(n+1)}(\Omega)$. This is accomplished by estimating some associated weighted integral $\int_\Omega u^{2(n+1)}\varphi(v)\,dx$ with a weight function $\varphi(v)$, which is uniformly bounded from above and below by positive constants. This approach was developed by Winkler in [26] for studying the chemotaxis system (1.1) with $f=0$, $g=1$ and χ from (1.2).

Lemma 2.3. *There exists a constant $c>0$ such that*:

$$\|u(.,t)\|_{L^{2(n+1)}(\Omega)} \leqslant c, \quad \text{for all } t \in (0, T_{\max}). \tag{2.2}$$

Proof. Set $p = 2(n+1)$, and fix $\kappa > 0$ small such that:

$$\kappa < \min\left\{\frac{p-1}{8p}, 2k-2\right\}, \tag{2.3}$$

where $k>1$ is the constant from (1.2). Then we pick $\eta>0$ large enough fulfilling:

$$\eta \geqslant \max\left\{\varrho\sqrt{\frac{2p(p-1)}{\kappa}}, \vartheta\right\}, \tag{2.4}$$

and define:

$$\varphi(s) = e^{(1+\eta s)^{-\kappa}}, \quad \text{for } s \geqslant 0.$$

By differentiation, using (1.1) and integration by parts, we obtain:

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\varphi(v)\,dx &= \int_\Omega u^{p-1}u_t\varphi(v)\,dx + \frac{1}{p}\int_\Omega u^p\varphi'(v)v_t\,dx \\
&= -(p-1)\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2\,dx - \int_\Omega u^{p-1}\varphi'(v)\nabla u\cdot\nabla v\,dx \\
&\quad + (p-1)\int_\Omega u^{p-1}\varphi(v)\chi(v)\nabla u\cdot\nabla v\,dx + \int_\Omega u^p\varphi'(v)\chi(v)|\nabla v|^2\,dx \\
&\quad + \int_\Omega u^{p-1}\varphi(v)f(u)\,dx - \int_\Omega u^{p-1}\varphi'(v)\nabla u\cdot\nabla v\,dx \\
&\quad - \frac{1}{p}\int_\Omega u^p\varphi''(v)|\nabla v|^2\,dx - \frac{1}{p}\int_\Omega u^p v\varphi'(v)\,dx + \frac{1}{p}\int_\Omega u^{p+1}\varphi'(v)g(u)\,dx.
\end{aligned}$$

By using (1.3), $\chi(s)\geqslant 0$, $g(s)\geqslant 0$ and $\varphi'(s)\leqslant 0$ for all $s\geqslant 0$, we get:

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\varphi(v)\,dx + (p-1)\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2\,dx + \frac{1}{p}\int_\Omega u^p\,\varphi''(v)|\nabla v|^2\,dx \\
&\quad\leqslant -2\int_\Omega u^{p-1}\varphi'(v)\nabla u\cdot\nabla v\,dx + (p-1)\int_\Omega u^{p-1}\varphi(v)\chi(v)\nabla u\cdot\nabla v\,dx \\
&\qquad - \frac{1}{p}\int_\Omega u^p v\varphi'(v)\,dx + a\int_\Omega u^p\varphi(v)\,dx.
\end{aligned} \tag{2.5}$$

Here, since

$$-s\varphi'(s) = \kappa\eta s(1+\eta s)^{-\kappa-1}e^{(1+\eta s)^{-\kappa}} \leqslant \kappa e^{(1+\eta s)^{-\kappa}} = \kappa\varphi(s),$$

for all $s\geqslant 0$, we obtain:

$$-\frac{1}{p}\int_{\Omega} u^p v\varphi'(v)\,\mathrm{d}x \leqslant \frac{\kappa}{p}\int_{\Omega} u^p\varphi(v)\,\mathrm{d}x. \tag{2.6}$$

We now make use of Young's inequality to the first and second terms on the right-hand side of (2.5) as follows:

$$-2\int_{\Omega} u^{p-1}\varphi'(v)\nabla u\cdot\nabla v\,\mathrm{d}x \leqslant \frac{p-1}{4}\int_{\Omega} u^{p-2}\varphi(v)|\nabla u|^2\,\mathrm{d}x + \frac{4}{(p-1)}\int_{\Omega} u^p\frac{\varphi'^2(v)}{\varphi(v)}|\nabla v|^2\,\mathrm{d}x, \tag{2.7}$$

and

$$\begin{aligned}
&(p-1)\int_{\Omega} u^{p-1}\varphi(v)\chi(v)\nabla u\cdot\nabla v\,\mathrm{d}x\\
&\quad\leqslant \frac{p-1}{4}\int_{\Omega} u^{p-2}\varphi(v)|\nabla u|^2\,\mathrm{d}x + (p-1)\int_{\Omega} u^p\varphi(v)\chi^2(v)|\nabla v|^2\,\mathrm{d}x\\
&\quad\leqslant \frac{p-1}{4}\int_{\Omega} u^{p-2}\varphi(v)|\nabla u|^2\,\mathrm{d}x + (p-1)\varrho^2\int_{\Omega} u^p\varphi(v)(1+\vartheta v)^{-2k}|\nabla v|^2\,\mathrm{d}x.
\end{aligned} \tag{2.8}$$

By inserting (2.6)–(2.8) into (2.5), we obtain:

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} u^p\varphi(v)\,\mathrm{d}x + \frac{p-1}{2}\int_{\Omega} u^{p-2}\varphi(v)|\nabla u|^2\,\mathrm{d}x + \frac{1}{p}\int_{\Omega} u^p\varphi''(v)|\nabla v|^2\,\mathrm{d}x\\
&\quad\leqslant \frac{4}{p-1}\int_{\Omega} u^p\frac{\varphi'^2(v)}{\varphi(v)}|\nabla v|^2\,\mathrm{d}x + (p-1)\varrho^2\int_{\Omega} u^p\varphi(v)(1+\vartheta v)^{-2k}|\nabla v|^2\,\mathrm{d}x + \left(\frac{\kappa}{p}+a\right)\int_{\Omega} u^p\varphi(v)\,\mathrm{d}x.
\end{aligned} \tag{2.9}$$

In the next step, we show that the terms on the right-hand side containing $|\nabla v|^2$ are dominated by $\frac{1}{p}\int_{\Omega} u^p\varphi''(v)|\nabla v|^2\,\mathrm{d}x$. In order to do this, we compute:

$$\begin{aligned}
I_1 &:= \frac{4}{p-1}\cdot\frac{\varphi'^2(s)}{\varphi(s)} = \frac{4}{p-1}\cdot\kappa^2\eta^2(1+\eta s)^{-2\kappa-2}\mathrm{e}^{(1+\eta s)^{-\kappa}},\\
I_2 &:= \varrho^2(p-1)(1+\vartheta s)^{-2k}\varphi(s) = \varrho^2(p-1)(1+\vartheta s)^{-2k}\mathrm{e}^{(1+\eta s)^{-\kappa}},\\
I_3 &:= \frac{1}{p}\varphi''(s) = \frac{1}{p}\eta^2\kappa(\kappa+1)(1+\eta s)^{-\kappa-2}\mathrm{e}^{(1+\eta s)^{-\kappa}} + \frac{1}{p}\kappa^2\eta^2(1+\eta s)^{-2\kappa-2}\mathrm{e}^{(1+\eta s)^{-\kappa}},
\end{aligned}$$

for $s \geqslant 0$. Hence,

$$\frac{I_1}{\frac{1}{2}I_3} \leqslant \frac{\frac{4}{p-1}\kappa^2\eta^2(1+\eta s)^{-2\kappa-2}\mathrm{e}^{(1+\eta s)^{-\kappa}}}{\frac{1}{2p}\eta^2\kappa(\kappa+1)(1+\eta s)^{-\kappa-2}\mathrm{e}^{(1+\eta s)^{-\kappa}}} = \frac{8p\kappa}{(p-1)(\kappa+1)}(1+\eta s)^{-\kappa} \leqslant \frac{8p\kappa}{p-1} \leqslant 1, \tag{2.10}$$

holds for $s \geqslant 0$ due to (2.3) and the fact that $\kappa > 0$. We also have:

$$\frac{I_2}{\frac{1}{2}I_3} \leqslant \frac{\varrho^2(p-1)(1+\vartheta s)^{-2k}\mathrm{e}^{(1+\eta s)^{-\kappa}}}{\frac{1}{2p}\eta^2\kappa(\kappa+1)(1+\eta s)^{-\kappa-2}\mathrm{e}^{(1+\eta s)^{-\kappa}}} = \frac{2\varrho^2p(p-1)}{\eta^2\kappa(\kappa+1)}(1+\vartheta s)^{-2k}(1+\eta s)^{\kappa+2}.$$

Now, we define:

$$\psi(s) = (1+\vartheta s)^{-2k}(1+\eta s)^{\kappa+2}.$$

Since $\kappa+2 < 2k$ by (2.3) and $\eta > \vartheta$, the function ψ satisfies $\psi'(s) \leqslant 0$ for all $s > 0$. Thus $\psi(s) \leqslant \psi(0) = 1$ for all $s \geqslant 0$. Therefore, in view of (2.4), we obtain:

$$\frac{I_2}{\frac{1}{2}I_3} \leqslant \frac{2p(p-1)\varrho^2}{\eta^2\kappa} \leqslant 1, \quad \text{for } s \geqslant 0. \tag{2.11}$$

From (2.10) and (2.11), we find that:

$$\frac{1}{p-1}\int_{\Omega} u^p\frac{\varphi'^2(v)}{\varphi(v)}|\nabla v|^2\,\mathrm{d}x + (p-1)\varrho^2\int_{\Omega} u^p\varphi(v)(1+\vartheta v)^{-2k}|\nabla v|^2\,\mathrm{d}x \leqslant \frac{1}{p}\int_{\Omega} u^p\varphi''(v)|\nabla v|^2\,\mathrm{d}x.$$

Hence by (2.9), we get:

$$\frac{d}{dt}\int_{\Omega} u^p\varphi(v)\,dx \leqslant \frac{-2(p-1)}{p}\int_{\Omega}\left|\nabla u^{p/2}\right|^2\varphi(v)\,dx + (\kappa + ap)\int_{\Omega} u^p\varphi(v)\,dx. \tag{2.12}$$

Now, adding $\frac{2(p-1)}{p}\int_{\Omega} u^p\varphi(v)\,dx$ in both sides of (2.12) gives:

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega} u^p\varphi(v)\,dx + \frac{2(p-1)}{p}\int_{\Omega} u^p\varphi(v)\,dx\\&\quad\leqslant \frac{-2(p-1)}{p}\int_{\Omega}\left|\nabla u^{p/2}\right|^2\varphi(v)\,dx + \left(\frac{2(p-1)}{p} + \kappa + ap\right)\int_{\Omega} u^p\varphi(v)\,dx.\end{aligned} \tag{2.13}$$

Next, we need the following known Gagliardo–Nirenberg inequality (see [9], for instance):

$$\|\omega\|_{L^q(\Omega)} \leqslant C_{GN}\|\omega\|^{\theta}_{W_2^1(\Omega)}\|\omega\|^{1-\theta}_{L^r(\Omega)}, \tag{2.14}$$

where

$$r \in (0,q), \quad \theta = \frac{\frac{n}{r} - \frac{n}{q}}{1 - \frac{n}{2} + \frac{n}{r}} \in (0,1).$$

We set $\alpha_0 = (\frac{2(p-1)}{p} + \kappa + ap)$ and applying the interpolation inequality (2.14) with $\omega = u^{p/2}$, $q = 2$, $r = \frac{2}{p}$ and $\theta = n(p-1)/(2-n+np)$, also using Young's inequality and (2.1), we obtain:

$$\begin{aligned}\alpha_0\int_{\Omega} u^p\varphi(v)\,dx &\leqslant \alpha_0 e\int_{\Omega} u^p\,dx = \alpha_0 e\left\|u^{p/2}\right\|^2_{L^2(\Omega)} \leqslant \alpha_0 e C_{GN}^2\left\|u^{p/2}\right\|^{2\theta}_{W_2^1(\Omega)}\left\|u^{p/2}\right\|^{2(1-\theta)}_{L^{2/p}(\Omega)}\\&\leqslant \frac{p-1}{p}\left\|u^{p/2}\right\|^2_{W_2^1(\Omega)} + c_1\left\|u^{p/2}\right\|^2_{L^{2/p}(\Omega)} = \frac{p-1}{p}\left\|u^{p/2}\right\|^2_{W_2^1(\Omega)} + c_1\|u\|^p_{L^1(\Omega)} \leqslant \frac{p-1}{p}\left\|u^{p/2}\right\|^2_{W_2^1(\Omega)} + c_2,\end{aligned} \tag{2.15}$$

where $c_1 = (\frac{\theta p}{p-1})^{\frac{\theta}{1-\theta}}(\alpha_0 e C_{GN}^2)^{\frac{1}{1-\theta}}(1-\theta)$ and $c_2 = (\max\{|\Omega|, \|u_0\|_{L^1(\Omega)}\}(\min\{1, \frac{b}{a}\})^{-1})^p c_1$. By inserting the last inequality in (2.13) and noting that $\varphi(s) \geqslant 1$ for all $s \geqslant 0$, we get:

$$\frac{d}{dt}\int_{\Omega} u^p\varphi(v)\,dx + \frac{p-1}{p}\int_{\Omega} u^p\varphi(v)\,dx \leqslant c_2.$$

Finally Gronwall's inequality yields:

$$\int_{\Omega} u^p\varphi(v)\,dx \leqslant c,$$

where c is some positive constant. This inequality along with $\varphi(s) \geqslant 1$ for all $s \geqslant 0$, yields (2.2). □

Lemma 2.4. *There exists a constant $c > 0$ such that the solution (u,v) of problem (1.1) satisfies*:

$$\left\|u(.,t)\right\|_{L^\infty(\Omega)} + \left\|v(.,t)\right\|_{L^\infty(\Omega)} \leqslant c, \quad \text{for all } t \in (0, T_{max}). \tag{2.16}$$

This lemma is shown in [13, Lemma 3.2].

Theorem 2.5. *The solution of (u,v) of the system (1.1) is global and bounded.*

Proof. First of all from the estimate (3.28) in [13], we have:

$$\left\|v(.,t)\right\|_{W^{1,q}(\Omega)} \leqslant c(\tau), \quad \text{for all } t \in (\tau, T_{max}), \tag{2.17}$$

where $q > n$ and $\tau \in (0, \min\{1, T_{max}\})$.

By considering the extensibility criterion provided by Lemma 2.1, the proof is a consequence of Lemma 2.4 and estimate (2.17). □

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