Global existence and boundedness of classical solutions for a chemotaxis model with logistic source

Existence globale et bornes des solutions classiques d'un modèle chimiotaxique avec une source logistique

Khadijeh Baghaei a, Mahmoud Hesaaraki b

a Department of Mathematics, Iran University of Science and Technology, Tehran, Iran
b Department of Mathematics, Sharif University of Technology, Tehran, Iran

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We consider the chemotaxis system:
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u \chi(v) \nabla v) + f(u), & x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} &= \Delta v - v + u g(u), & x \in \Omega, \ t > 0,
\end{aligned}
\]

under homogeneous Neumann boundary conditions in a bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 1 \), with smooth boundary and function \( f \) is assumed to generalize the logistic source:
\[
f(u) = au - bu^2, \ u \geq 0, \text{ with } a > 0, \ b > 0.
\]

Moreover, \( \chi(s) \) and \( g(s) \) are nonnegative smooth functions and satisfy:
\[
\chi(s) \leq \frac{\varrho}{(1 + \vartheta s)^k}, \ s \geq 0, \text{ with some } \varrho > 0, \ \vartheta > 0 \text{ and } k > 1.
\]
\[
g(s) \leq \frac{h_0}{(1 + h s)^\delta}, \ s \geq 0, \text{ with } h_0 > 0, \ h > 0, \ \delta > 0.
\]

We prove that for all positive values of \( \varrho, a \) and \( b \), classical solutions to the above system are uniformly-in-time bounded. This result extends a recent result by C. Mu, L. Wang, P. Zheng and Q. Zhang (2013) [13], which asserts the global existence and boundedness of classical solutions on condition that \( 0 \leq a < 2b \) and \( \varrho \) be sufficiently small.

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In this paper, we investigate the global existence and boundedness of solutions to the parabolic system:

\[
\begin{aligned}
&u_t = \Delta u - \nabla \cdot (u \chi(v)\nabla v) + f(u), \quad x \in \Omega, \ t > 0, \\
v_t = \Delta v - v + u g(u), \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_0, \quad v(x, 0) = v_0, \quad x \in \Omega,
\end{aligned}
\]  

(1.1)

in a smooth bounded domain \( \Omega \subseteq \mathbb{R}^n, n \geq 1 \), where \( v \) denotes the unit outward normal vector to \( \partial \Omega \) and \( u_0 \) and \( v_0 \) are given nonnegative functions. Moreover, the functions \( \chi(s), f(s) \) and \( g(s) \) are smooth and \( \chi(s) \) and \( g(s) \) are nonnegative for \( s \geq 0 \).

In mathematical biology, systems like (1.1) are used to describe the movement of cells towards higher concentration of a chemical signal substance produced by the cells themselves. Such a movement is called chemotaxis. Here, \( \chi(\cdot) \) is the chemotactic sensitivity function, \( f(\cdot) \) is the growth of \( u \) and \( u \) is the production of \( v \) [5,11,12].

Several types of functions have been proposed for \( \chi(v) \), \( f(u) \) and \( g(u) \) (see [5,17,18,25]). For \( f = 0 \) and \( \chi \) and \( g \) constants: it is known that when \( n = 1 \), the blow-up never can occur [19]. For \( n = 2 \), it has been shown in [16] that if \( \|u_0\|_{L^1(\Omega)} < \frac{4\pi}{16\pi^2} \), then all solutions of (1.1) are global and bounded. The same result holds for radial solutions when \( \|u_0\|_{L^1(\Omega)} < \frac{8\pi}{16\pi^2} \) [16]. While for \( \|u_0\|_{L^1(\Omega)} > \frac{4\pi}{16\pi^2} \), solutions blow up either in finite or infinite time [8], moreover, the authors in [3] constructed radial solutions to (1.1) which blow up in finite time provided that \( \|u_0\|_{L^1(\Omega)} > \frac{8\pi}{16\pi^2} \). For \( n \geq 3 \), Winkler [27] proved that, for each \( q > \frac{n}{2} \) and \( p > n \), there exists \( \epsilon_0 > 0 \) such that, if \( \|u_0\|_{L^1(\Omega)} < \epsilon \) and \( \|\nabla v_0\|_{L^p(\Omega)} < \epsilon \) for \( \epsilon < \epsilon_0 \), then solutions of (1.1) are global and bounded. Also, when \( \Omega \) is a ball in \( \mathbb{R}^n \), \( n \geq 3 \), the same author in [29] showed that, under some conditions on initial data and provided that \( \|u_0\|_{L^1(\Omega)} > 0 \), there exist radial solutions that blow up in finite time.

In [18], the authors proved the global existence of solutions of (1.1) for \( n = 2 \), \( g = \text{constant} \), \( \chi \) a smooth bounded function and:

\[ f(u) = au - bu^2, \quad u \geq 0, \]  

with \( -\infty < a < \infty, \ b > 0 \) and \( f(0) = 0 \). The authors in [17] proved the global existence of solutions of (1.1) in the two-dimensional case under the assumptions that \( \chi \) is a positive constant and the functions \( f(u) \) and \( g(u) \) are smooth, and:

\[ f(u) \leq 1 - bu^\alpha, \quad u \geq 0, \ b > 0, \ \alpha \geq 1, \]  

\[ 0 \leq h(u) \leq (u + 1)^\beta, \quad 0 \leq h'(u) \leq \beta(u + 1)^{\beta - 1}, \quad u \geq 0, \ \beta > 0, \]

where \( h(u) = ug(u) \). These results depend on the values of \( \alpha \) and \( \beta \). For boundedness of solutions, they showed that:

\[ \|u(t)\|_{L^1(\Omega)} + \|v(t)\|_{H^2(\Omega)} \leq \Psi(\|u_0\|_{L^1(\Omega)} + \|v_0\|_{H^2(\Omega)}), \quad t \geq 0, \]

with some increasing function \( \psi \).

For \( n \geq 3 \), Winkler in [25] proved that if \( \chi \in \mathbb{R} \) constant, \( g = 1 \), \( \Omega \) a bounded convex domain of \( \mathbb{R}^n \) with smooth boundary and:

\[ f(u) \leq a - bu^2, \quad u \geq 0, \]  

with \( a > 0, \ b > 0 \) and \( f(0) \geq 0 \), then problem (1.1) admits a unique global classical solution, which is bounded in \( \Omega \times (0, \infty) \) provided that \( b \) is sufficiently large.
There are much more works related to chemotaxis system in the literature; we refer the interested readers to [1,2,4,6,7,9,10,14,15,20–24,28,30] and the references therein.

Throughout this paper we assume that \(\chi(v), f(u)\) and \(g(u)\) satisfy the following conditions:

\[
\chi(v) \leq \frac{\vartheta}{1 + \vartheta v}, \quad v \geq 0, \text{ and some } \vartheta > 0, \varphi > 0 \text{ and } k > 1, \tag{1.2}
\]

\[
f(u) \leq au - bu^2, \quad u \geq 0, \text{ with } a > 0, b > 0 \text{ and } f(0) = 0, \tag{1.3}
\]

\[
g(u) \leq \frac{h_0}{(1 + hu)^\delta}, \quad u \geq 0, \text{ with } h_0 > 0, h > 0, \delta > 0. \tag{1.4}
\]

Winkler in [26] proved that if \(f = 0, g = 1\) and \(\chi\) satisfies the condition (1.2), then problem (1.1) has a unique global classical solution, which is uniformly bounded.

In [13], the authors showed that the solutions of problem (1.1) with (1.2), (1.3) and (1.4) are global and bounded provided that \(\varphi\) and \(a\) are sufficiently small. They left open the question of whether there exists a blow-up solution to problem (1.1), when the parameter \(a\) is sufficiently large. We answer this question and prove that solutions of problem (1.1) are global and bounded for all positive values of \(\varphi, a\) and \(b\). This means that the blow-up never can occur for all positive values of \(\varphi, a\) and \(b\).

2. Global existence

Let us state the standard well-posedness and classical solvability result.

**Lemma 2.1.** Let the nonnegative functions \(u_0\) and \(v_0\) satisfy \(u_0 \in C^0(\overline{\Omega})\) and \(v_0 \in W^{1,q}(\Omega)\) for some \(q > n\). Then problem (1.1) has a unique local-in-time nonnegative classical solution:

\[
u \in C^0(\overline{\Omega} \times [0, \max]) \cap C^2(\overline{\Omega} \times (0, \max)),
\]

\[
u \in C^0(\overline{\Omega} \times [0, \max]) \cap C^2(\overline{\Omega} \times (0, \max)) \cap L^\infty_{\text{loc}}([0, \max); W^{1,q}(\Omega)),
\]

where \(\max\) denotes the maximal existence time. In addition, if \(\max < +\infty\), then:

\[\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,q}(\Omega)} \to \infty \text{ as } t \nearrow \max.\]

The proof follows from the general theory of parabolic systems. For details of the proof, we refer the reader to [9,26].

**Lemma 2.2.** Suppose that \(f\) satisfies (1.3) with some \(a, b > 0\). Then we have:

\[\|u(t)\|_{L^1(\Omega)} \leq \frac{\max\{|\Omega|, \|u_0\|_{L^1(\Omega)}\}}{\min\{1, \frac{b}{a}\}}. \tag{2.1}\]

**Proof.** We integrate the first equation in (1.1) over \(\Omega\) and using (1.3) to see that:

\[\frac{d}{dt} \int_{\Omega} u(x, t) \, dx = \int_{\Omega} f(u) \, dx \leq a \int_{\Omega} u \, dx - b \int_{\Omega} u^2 \, dx, \quad \text{for } t > 0.
\]

From the Hölder inequality, we obtain \(\int_{\Omega} u \, dx \leq |\Omega|^{1/2} \int_{\Omega} u^2 \, dx\), and hence \(y(t) = \int_{\Omega} u(x, t) \, dx\) satisfies:

\[y'(t) \leq ay(t) - b|\Omega|^{-1}y^2(t), \quad \text{for } t > 0.
\]

Now, we set \(\gamma(t) := y(t)^{-1}\), thus we obtain:

\[\gamma'(t) + a\gamma(t) \geq b|\Omega|^{-1},
\]

which yields:

\[\gamma(t) \geq e^{-at}\gamma(0) + \frac{b}{a}|\Omega|^{-1}(1 - e^{-at}).
\]

Therefore,

\[y(t) \leq \left(\gamma(0)^{-1}e^{-at} + \frac{b}{a}|\Omega|^{-1}(1 - e^{-at})\right)^{-1}.
\]

This inequality yields:
studying the chemotaxis system (1.1) with $y(t) \leq \frac{\max\{|\Omega|, y(0)|}{\min\{1, b_0\}}$.

So, the proof is complete. □

The main step towards the global existence of solutions is to establish a uniform bound of $u(., t)$ in the space $L^{2(n+1)}(\Omega)$. This is accomplished by estimating some associated weighted integral $\int_{\Omega} u^{2(n+1)} \varphi(v) \, dx$ with a weight function $\varphi(v)$, which is uniformly bounded from above and below by positive constants. This approach was developed by Winkler in [26] for studying the chemotaxis system (1.1) with $f = 0$, $g = 1$ and $\chi$ from (1.2).

**Lemma 2.3.** There exists a constant $c > 0$ such that:

$$\left\|u(., t)\right\|_{L^{2(n+1)}(\Omega)} \leq c, \quad \text{for all } t \in (0, T_{\max}).$$

**(Proof.)** Set $p = 2(n+1)$, and fix $\kappa > 0$ small such that:

$$\kappa < \min\left\{\frac{p - 1}{8p}, 2k - 2\right\},$$

where $k > 1$ is the constant from (1.2). Then we pick $\eta > 0$ large enough fulfilling:

$$\eta \geq \max\left\{\sqrt{\frac{2p(p - 1)}{\kappa}}, \vartheta\right\},$$

and define:

$$\varphi(s) = e^{(1 + \eta s)^{-\kappa}}, \quad \text{for } s \geq 0.$$  

By differentiation, using (1.1) and integration by parts, we obtain:

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \varphi(v) \, dx = \int_{\Omega} u^{p-1} u_t \varphi(v) \, dx + \frac{1}{p} \int_{\Omega} u^p \varphi'(v) v_t \, dx$$

$$= -(p - 1) \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \, dx - \int_{\Omega} u^{p-1} \varphi'(v) \nabla u \cdot \nabla v \, dx$$

$$+ (p - 1) \int_{\Omega} u^{p-1} \varphi(v) \chi(v) \nabla u \cdot \nabla v \, dx + \int_{\Omega} u^p \varphi'(v) \chi(v) |\nabla v|^2 \, dx$$

$$+ \int_{\Omega} u^{p-1} \varphi(v) f(u) \, dx - \int_{\Omega} u^{p-1} \varphi'(v) \nabla u \cdot \nabla v \, dx$$

$$- \frac{1}{p} \int_{\Omega} u^{p-1} \varphi''(v) |\nabla v|^2 \, dx - \frac{1}{p} \int_{\Omega} u^p \varphi'(v) v \, dx + \frac{1}{p} \int_{\Omega} u^{p+1} \varphi'(v) g(u) \, dx.$$

By using (1.3), $\chi(s) \geq 0$, $g(s) \geq 0$ and $\varphi'(s) \leq 0$ for all $s \geq 0$, we get:

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \varphi(v) \, dx + (p - 1) \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \, dx + \frac{1}{p} \int_{\Omega} u^p \varphi''(v) |\nabla v|^2 \, dx$$

$$\leq -2 \int_{\Omega} u^{p-1} \varphi'(v) |\nabla u \cdot \nabla v| \, dx + (p - 1) \int_{\Omega} u^{p-1} \varphi(v) \chi(v) |\nabla v|^2 \, dx$$

$$- \frac{1}{p} \int_{\Omega} u^p \varphi'(v) v \, dx + a \int_{\Omega} u^p \varphi(v) \, dx.$$  

Here, since

$$-s \varphi'(s) = \kappa \eta s (1 + \eta s)^{-\kappa - 1} e^{(1 + \eta s)^{-\kappa}} \leq \kappa e^{(1 + \eta s)^{-\kappa}} = \kappa \varphi(s),$$

for all $s \geq 0$, we obtain:
In order to do this, we compute:

\[- \frac{1}{p} \int_{\Omega} u^p v' (v) \, dx \leq \frac{\kappa}{p} \int_{\Omega} u^p \varphi(v) \, dx. \quad (2.6)\]

We now make use of Young's inequality to the first and second terms on the right-hand side of (2.5) as follows:

\[-2 \int_{\Omega} u^{p-1} \varphi'(v) \nabla u \cdot \nabla v \, dx \leq \frac{p-1}{4} \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \, dx + \frac{4}{(p-1)} \int_{\Omega} u^{p} \frac{\varphi''(v)}{\varphi(v)} |\nabla v|^2 \, dx, \quad (2.7)\]

and

\[(p-1) \int_{\Omega} u^{p-1} \varphi(v) \chi(v) \nabla u \cdot \nabla v \, dx \leq \frac{p-1}{4} \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \, dx + (p-1) \int_{\Omega} u^{p} \varphi(v) \chi^2(v) |\nabla v|^2 \, dx \leq \frac{p-1}{4} \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \, dx + (p-1) \int_{\Omega} u^{p} \varphi(v) (1 + \vartheta v)^{-2k} |\nabla v|^2 \, dx. \quad (2.8)\]

By inserting (2.6)–(2.8) into (2.5), we obtain:

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \varphi(v) \, dx + \frac{p-1}{2} \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \, dx + \frac{1}{p} \int_{\Omega} u^{p} \varphi''(v) |\nabla v|^2 \, dx \leq \frac{4}{p-1} \int_{\Omega} u^{p} \varphi(v) (1 + \vartheta v)^{-2k} |\nabla v|^2 \, dx + \left( \frac{\kappa}{p} + a \right) \int_{\Omega} u^{p} \varphi(v) \, dx. \quad (2.9)\]

In the next step, we show that the terms on the right-hand side containing $|\nabla v|^2$ are dominated by $\frac{1}{p} \int_{\Omega} u^p \varphi''(v) |\nabla v|^2 \, dx$. In order to do this, we compute:

\[
I_1 := \frac{4}{p-1} \frac{\varphi''(s)}{\varphi(s)} = \frac{4}{p-1} \kappa^2 \eta^2 (1 + \eta s)^{-2k-2} e^{(1+\eta s)^{-k}},
I_2 := \varphi^2 (p-1) (1 + \vartheta s)^{-2k} \varphi(s) = \varphi^2 (p-1) (1 + \vartheta s)^{-2k} e^{(1+\eta s)^{-k}},
I_3 := \frac{1}{p} \varphi''(s) = \frac{1}{p} \eta^2 \kappa (k+1) (1 + \eta s)^{-k-2} e^{(1+\eta s)^{-k}} + \frac{1}{p} \kappa^2 \eta^2 (1 + \eta s)^{-2k-2} e^{(1+\eta s)^{-k}},
\]

for $s \geq 0$. Hence,

\[
\frac{I_1}{I_3} \leq \frac{8 \rho \kappa}{(p-1)(k+1)} \eta^2 \kappa (k+1) (1 + \eta s)^{-k-2} e^{(1+\eta s)^{-k}} \leq \frac{8 \rho \kappa}{p-1} \leq 1, \quad (2.10)
\]

holds for $s \geq 0$ due to (2.3) and the fact that $\kappa > 0$. We also have:

\[
\frac{I_2}{I_3} \leq \frac{2 \rho^2 p (p-1)}{\eta^2 \kappa (k+1) (1 + \eta s)^{k+2}} \leq \frac{2 \rho^2 p (p-1)}{\eta^2 \kappa (k+1) (1 + \eta s)^{k+2}}.
\]

Now, we define:

\[
\psi(s) = (1 + \vartheta s)^{-2k(1 + \eta s)^{k+2}}.
\]

Since $\kappa + 2k < 2k$ by (2.3) and $\eta > \vartheta$, the function $\psi$ satisfies $\psi(s) \leq 0$ for all $s > 0$. Thus $\psi(s) \leq \psi(0) = 1$ for all $s \geq 0$. Therefore, in view of (2.4), we obtain:

\[
\frac{I_2}{I_3} \leq \frac{2 \rho (p-1) \rho^2}{\eta^2 k} \leq 1, \quad \text{for } s \geq 0. \quad (2.11)
\]

From (2.10) and (2.11), we find that:

\[
\frac{1}{p-1} \int_{\Omega} u^p \frac{\varphi''(v)}{\varphi(v)} |\nabla v|^2 \, dx + (p-1) \int_{\Omega} u^p \varphi(v) (1 + \vartheta v)^{-2k} |\nabla v|^2 \, dx \leq \frac{1}{p} \int_{\Omega} u^p \varphi''(v) |\nabla v|^2 \, dx.
\]
Hence by (2.9), we get:
\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \, dx \leq -\frac{2(p-1)}{p} \int_{\Omega} |\nabla u|^{p/2} \varphi(v) \, dx + (\kappa + ap) \int_{\Omega} u^p \varphi(v) \, dx.
\] (2.12)

Now, adding \(\frac{2(p-1)}{p} \int_{\Omega} u^p \varphi(v) \, dx\) in both sides of (2.12) gives:
\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \, dx + \frac{2(p-1)}{p} \int_{\Omega} u^p \varphi(v) \, dx \\
\leq \frac{-2(p-1)}{p} \int_{\Omega} |\nabla u|^{p/2} \varphi(v) \, dx + \left(\frac{2(p-1)}{p} + \kappa + ap\right) \int_{\Omega} u^p \varphi(v) \, dx.
\] (2.13)

Next, we need the following known Gagliardo–Nirenberg inequality (see [9], for instance):
\[
\|\omega\|_{L^q(\Omega)} \leq C_{GN} \|\omega\|^\theta_{W^1_p(\Omega)} \|\omega\|^{1-\theta}_{L^\infty(\Omega)},
\] (2.14)

where
\[
r \in (0, q), \quad \theta = \frac{n \tau - \frac{n}{2} + \frac{p}{\tau}}{1 - \frac{2}{p} + \frac{p}{\tau}} \in (0, 1).
\]

We set \(\alpha_0 = (\frac{2(p-1)}{p} + \kappa + ap)\) and applying the interpolation inequality (2.14) with \(\omega = u^p/2, q = 2, r = \frac{2}{p}\) and \(\theta = \frac{n(p - 1)}{2 - n + np}\), also using Young's inequality and (21), we obtain:
\[
\alpha_0 \int_{\Omega} u^p \varphi(v) \, dx = \alpha_0 e \int_{\Omega} u^p \, dx = \alpha_0 e \int_{\Omega} u^{p/2} \|\nabla u\|^{p/2}_{L^2(\Omega)} \leq \alpha_0 e C_{GN}^2 \|u^{p/2}\|_{W^1_{2} L^2(\Omega)}^2 \|u^{p/2}\|_{L^2(\Omega)}^{2(1-\theta)}
\leq \frac{p-1}{p} \|u^{p/2}\|_{W^1_{2} L^2(\Omega)}^2 + c_1 \|u^{p/2}\|_{L^2(\Omega)}^2 = \frac{p-1}{p} \|u^{p/2}\|_{W^1_{2} L^2(\Omega)}^2 + c_1 \|u^{p/2}\|_{L^1(\Omega)}^2 \leq \frac{p-1}{p} \|u^{p/2}\|_{L^1(\Omega)}^2 \leq \frac{p-1}{p} \|u^{p/2}\|_{L^1(\Omega)}^2 + c_2.
\] (2.15)

where \(c_1 = (\frac{2(p-1)}{p})^{\frac{\theta}{p}} (\alpha_0 e C_{GN}^2)^{\frac{1}{p}} (1-\theta)\) and \(c_2 = (\max(|\Omega|, \|u_0\|_{L^1(\Omega)}) (\min(1, \frac{b}{\theta}))^{-1})^p c_1\). By inserting the last inequality in (2.13) and noting that \(\varphi(s) \geq 1\) for all \(s \geq 0\), we get:
\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \, dx + \frac{p-1}{p} \int_{\Omega} u^p \varphi(v) \, dx \leq c_2.
\]

Finally Gronwall's inequality yields:
\[
\int_{\Omega} u^p \varphi(v) \, dx \leq c,
\]
where \(c\) is some positive constant. This inequality along with \(\varphi(s) \geq 1\) for all \(s \geq 0\), yields (2.2). \(\square\)

**Lemma 2.4.** There exists a constant \(c > 0\) such that the solution \((u, v)\) of problem (1.1) satisfies:
\[
\|u(., t)\|_{L^\infty(\Omega)} + \|v(., t)\|_{L^\infty(\Omega)} \leq c, \quad \text{for all } t \in (0, T_{\max}).
\] (2.16)

This lemma is shown in [13, Lemma 3.2].

**Theorem 2.5.** The solution of \((u, v)\) of the system (1.1) is global and bounded.

**Proof.** First of all from the estimate (3.28) in [13], we have:
\[
\|v(., t)\|_{W^{1, q}(\Omega)} \leq c(t), \quad \text{for all } t \in (\tau, T_{\max}),
\] (2.17)

where \(q > n\) and \(\tau \in (0, \min(1, T_{\max}))\).

By considering the extensibility criterion provided by Lemma 2.1, the proof is a consequence of Lemma 2.4 and estimate (2.17). \(\square\)
References


