C. R. Acad. Sci. Paris, Ser. I 351 (2013) 747–752

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Numerical analysis

Cauchy problems with modified conditions for the Euler–Poisson–Darboux equations in the hyperbolic space

Problèmes de Cauchy avec des conditions modifiées pour les équations d'Euler–Poisson–Darboux dans l'espace hyperbolique

Cheikh Ould Mohamed El-Hafedh, Elbar Ould Ely Telmoudy, Mohamed Vall Ould Moustapha

Unité de recherche "Analyse, EDP et modélisation", faculté des sciences et techniques (FST), université des sciences, de technologie et de la medécine (USTM), B.P. 5026, Mauritania

ARTICLE INFO

Article history:
Received 2 May 2013
Accepted after revision 31 July 2013
Available online 23 October 2013

Presented by the Editorial Board

ABSTRACT

In this note, we give the solutions of the Cauchy problems for the Euler–Poisson–Darboux equations (EPD) with modified conditions in the hyperbolic space with application to the wave equation.

RÉSUMÉ

On donne les solutions explicites des problèmes de Cauchy pour les équations d'Euler–Poisson–Darboux, avec des conditions modifiées dans l'espace hyperbolique avec application à l'équation des ondes.

Version française abrégée

Dans [2], le premier et le troisième auteur ont obtenu les solutions explicites des problèmes de Cauchy avec des conditions modifiées pour les équations d'Euler–Poisson–Darboux dans l'espace euclidien. Dans ce travail nous donnons les solutions explicites des problèmes de Cauchy avec des conditions modifiées pour les équations d'Euler–Poisson–Darboux dans l'espace hyperbolique. Noter que le problème de Cauchy classique pour l'équation d'Euler–Poisson–Darboux dans l'espace hyperbolique est considéré dans [4] et [5] :

$$\begin{cases} (a) \quad L_n U(t,x) = A_t^{\mu} U(t,x), \quad 0<t,\ x\in\mathbb{H}^n \\ (b)'' \quad U(0,x) = f(x), \quad \partial_t U(0,x) = 0; \quad f\in C^{\infty}(\mathbb{H}^n). \end{cases} \tag{$(E_\mu^n)''$}$$

Plus explicitement, nous nous intéressons à la famille de problèmes :

$$\begin{cases} (a) \quad L_n U(t,x) = A_t^{\mu} U(t,x), \quad 0<t,\ x\in\mathbb{H}^n \\ (b) \quad U(0,x) = f(x), \quad \lim_{t\to 0} t^{1-2\mu}\partial_t U(t,x) = g(x); \quad f,g\in C^{\infty}(\mathbb{H}^n) \end{cases} \tag{E_μ^n}$$

E-mail addresses: cheikh976@yahoo.fr (C.O.M. El-Hafedh), elbar@nasr.mr (E.O.E. Telmoudy), mohamedvall.ouldmoustapha230@gmail.com (M.V. Ould Moustapha).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.07.026

$$\begin{cases} (a)' \quad A_x^\nu U(t,x) = A_t^\mu U(t,x), \quad 0<t,\ 0<x \\ (b)' \quad U(0,x) = f(x), \quad \lim_{t\to 0} t^{1-2\mu}\partial_t U(t,x) = g(x); \quad f,g\in C^\infty(\mathbb{R}_+) \end{cases} \qquad (E_\mu^\nu)'$$

où L_n est l'opérateur de Laplace–Beltrami associé à l'espace Riemannien hyperbolique $\mathbb{H}^n$, donné en coordonnées géodésiques polaires par :

$$L_n = \frac{\partial^2}{\partial r^2} + (n-1)\coth r\frac{\partial}{\partial r} + \left(\frac{n-1}{2}\right)^2 + \Lambda(r) \tag{0.1}$$

avec $\Lambda(r)$ un opérateur differentiel du second ordre sur la sphère $\mathbb{S}^{n-1}(r)$ de rayon r. L'opérateur A_x^ν est donné par :

$$A_x^\nu := \frac{\partial^2}{\partial x^2} + (1-2\nu)\coth x\frac{\partial}{\partial x} + \left(\frac{1-2\nu}{2}\right)^2. \tag{0.2}$$

Remarquons que, dans $(E_\mu^n)''$, la deuxième donnée est nulle ($g=0$), car une solution de l'équation (a) ne saurait être régulière pour $t=0$ que si sa dérivée première par rapport à t s'y annule. Les conditions modifiées (b) et $(b)'$ permettent de prendre la deuxième donnée comme une fonction quelconque g, nulle ou non, tout en recouvrant les conditions de Cauchy classiques $(b)''$. Ainsi, les problèmes de Cauchy $(E_{\frac{1}{2}}^n)$ et $(E_{\frac{1}{2}}^\nu)'$ correspondent respectivement aux équations classique (voir [3], [7] et [1]) et radiale (Théorème 2) des ondes dans $\mathbb{H}^n$. Les résultats principaux de cet article – Théorèmes 1, 2 et 3 – sont donnés dans l'introduction et leurs applications sont dans la section 6.

1. Introduction

This work is motivated by the paper [2], in which are formulated and solved the Cauchy problems with modified conditions for the classical and radial Euler–Poisson–Darboux equations in the Euclidean space. The aim of this paper is to formulate and to discuss the analogous modified Cauchy problems for the Euler–Poisson–Darboux equations in the hyperbolic space. The classical Cauchy problem associated with the Euler–Poisson–Darboux equation $(E_\mu^n)''$ has been studied in [4] and [5]. Therefore, we generalize and unify several results of this equation. The obtained results are applied to the classical wave equation on $\mathbb{H}^n$ (see [3], [7] and [1]). The main results in this note are as following:

Theorem 1 *(Classical EPD with modified initial conditions). Let $\mu\in(0,\frac{1}{2})$. The Cauchy problem (E_μ^n) with modified conditions for the classical Euler–Poisson–Darboux equation on the hyperbolic space has the unique solution given by*:

$$U(t,x) = \alpha_{n,-\mu}(\sinh t)^{2\mu}\left(\frac{\partial}{\sinh t\partial t}\right)^{\frac{n-1}{2}} \int\limits_{r<t} f(x')\left(\sinh^2\frac{t}{2} - \sinh^2\frac{r}{2}\right)^{-\mu-\frac{1}{2}} d\mu(x')$$
$$+ \frac{1}{2\mu}\alpha_{n,\mu}\left(\frac{\partial}{\sinh t\partial t}\right)^{\frac{n-1}{2}} \int\limits_{r<t} g(x')\left(\sinh^2\frac{t}{2} - \sinh^2\frac{r}{2}\right)^{\mu-\frac{1}{2}} d\mu(x')$$

when n is odd,

$$U(t,x) = \beta_{n,-\mu}(\sinh t)^{2\mu}\left(\frac{\partial}{\sinh t\partial t}\right)^{\frac{n}{2}} \int\limits_{r<t} f(x')\left(\sinh^2\frac{t}{2} - \sinh^2\frac{r}{2}\right)^{-\mu} d\mu(x')$$
$$+ \frac{1}{2\mu}\beta_{n,\mu}\left(\frac{\partial}{\sinh t\partial t}\right)^{\frac{n}{2}} \int\limits_{r<t} g(x')\left(\sinh^2\frac{t}{2} - \sinh^2\frac{r}{2}\right)^{\mu} d\mu(x')$$

when n is even, where

$$\alpha_{n,\mu} = \frac{1}{2}\frac{\Gamma(1+2\mu)}{(2\pi)^{\frac{n-1}{2}}\Gamma^2(\frac{1}{2}+\mu)}, \qquad \beta_{n,\mu} = \frac{4^\mu}{(2\pi)^{\frac{n}{2}}}$$

and $r=d(x,x')$ is the geodesic distance between x and x' in $\mathbb{H}^n$.

Theorem 2 *(Radial wave equation). Let $\nu<\frac{1}{2}$. The Cauchy problem $(E_{\frac{1}{2}}^\nu)'$ for the radial wave equation on the hyperbolic space has the unique solution given by*:

$$U(t,x) = \int_0^{+\infty} f(x')\frac{\partial}{\partial t}W(t,x,x')(\sinh x')^{1-2\nu}\,dx' + \int_0^{+\infty} g(x')W(t,x,x')(\sinh x')^{1-2\nu}\,dx'$$

where

$$W\left(t,x,x'\right)=4^{\nu-1}\left(\sinh\frac{x}{2}\sinh\frac{x'}{2}\right)^{\nu}\left(\cosh\frac{x'}{2}\right)^{2\nu}$$
$$\times\int_0^{+\infty}\int_{|\ln\frac{y}{y'}|<\frac{t}{2}} J_{-\nu}\left(y\sinh\frac{x}{2}\right)J_{-\nu}\left(y'\sinh\frac{x'}{2}\right)J_0\left(\sqrt{2yy'\cosh\frac{t}{2}-y^2-y'^2}\right)y^{-\nu}y'^{\nu}\,dy\,dy' \tag{1.1}$$

and J_ν is the Bessel function defined by [8, p. 65].

Theorem 3 *(Radial EPD with modified initial conditions). Let $\nu<\frac{1}{2}$ and $\mu\in(0,\frac{1}{2})$ be given. The Cauchy problem $(E^\nu_\mu)'$ with modified conditions for the radial Euler–Poisson–Darboux equation on the hyperbolic space has the unique solution given by*:

$$U(t,x)=(\sinh t)^{2\mu}\int_0^{+\infty} f\left(x'\right)W_{-\mu}\left(t,x,x'\right)\left(\sinh x'\right)^{1-2\nu}dx'+\frac{1}{2\mu}\int_0^{+\infty} g\left(x'\right)W_{\mu}\left(t,x,x'\right)\left(\sinh x'\right)^{1-2\nu}dx'$$

where

$$W_\mu\left(t,x,x'\right)=4^{\nu+\mu-1}\frac{\Gamma(1+\mu)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\mu)}\left(\sinh\frac{x}{2}\sinh\frac{x'}{2}\right)^{\nu}\left(\cosh\frac{x'}{2}\right)^{2\nu}\int_0^t\left(\sinh^2\frac{t}{2}-\sinh^2\frac{z}{2}\right)^{\mu-\frac{1}{2}}$$
$$\times\frac{\partial}{\partial z}\int_0^{+\infty}\int_{|\ln\frac{y}{y'}|<\frac{z}{2}} J_{-\nu}\left(y\sinh\frac{x}{2}\right)J_{-\nu}\left(y'\sinh\frac{x'}{2}\right)J_0\left(\sqrt{2yy'\cosh\frac{z}{2}-y^2-y'^2}\right)y^{-\nu}y'^{\nu}\,dy\,dy'\,dz. \tag{1.2}$$

2. Preliminaries

We recall the Jacobi transform related directly to that studied by Koornwinder [6]:

$$\widehat{f}(\lambda)=\int_0^{+\infty} f(x)\Phi^\nu_\lambda(x)(2\sinh x)^{1-2\nu}dx,\quad \Phi^\nu_\lambda(x)={}_2F_1\left(\frac{\frac{1}{2}-\nu-\imath\lambda}{2},\frac{\frac{1}{2}-\nu+\imath\lambda}{2},1-\nu,-\sinh^2 x\right). \tag{2.1}$$

The function $\Phi^\nu_\lambda(x)$ is an eigenfunction of the operator A^ν_x (given in (0.2)) associated with the value $-\lambda^2$. For $\nu<\frac{1}{2}$, an inverse transform of the Jacobi transform is given by:

$$f(x)=\frac{1}{2\pi}\int_0^{+\infty}\widehat{f}(\lambda)\Phi^\nu_\lambda(x)\left|C_\nu(\lambda)\right|^{-2}d\lambda,\quad C_\nu(\lambda)=\frac{2^{\frac{1}{2}-\nu-\imath\lambda}\Gamma(1-\nu)\Gamma(\imath\lambda)}{\Gamma(\frac{\frac{1}{2}-\nu+\imath\lambda}{2})\Gamma(\frac{\frac{3}{2}-\nu+\imath\lambda}{2})}. \tag{2.2}$$

Lemma 1. *We have $\widehat{A^\nu_x f}(\lambda)=-\lambda^2\widehat{f}(\lambda),\ \lambda\in\mathbb{R}^+$.*

Proof. It suffices to write:

$$A^\nu_x=\frac{1}{(\sinh x)^{1-2\nu}}\frac{\partial}{\partial x}(\sinh x)^{1-2\nu}\frac{\partial}{\partial x}+\left(\frac{1-2\nu}{2}\right)^2,$$

and to do two integrations by parts. □

Lemma 2. *For $t>0$ and $x,x'>0$ let*:

$$J\left(t,x,x'\right)=\int_0^{+\infty}\int_{|\ln\frac{y}{y'}|<\frac{t}{2}} J_{-\nu}\left(y\sinh\frac{x}{2}\right)J_{-\nu}\left(y'\sinh\frac{x'}{2}\right)J_0\left(\sqrt{2yy'\cosh\frac{t}{2}-y^2-y'^2}\right)y^{-\nu}y'^{\nu}\,dy\,dy'.$$

Then if t is sufficiently small, we have the following asymptotic formula:

$$J(t,x,x') \approx \frac{\sinh\frac{t}{2}}{2\pi\sinh\frac{x}{2}\sinh\frac{x'}{2}}\int_{-1}^{1}\frac{1}{\sqrt{z}}{}_2F_1\left(\frac{1}{2}-\nu,\frac{1}{2}+\nu,\frac{1}{2},z\right)\mathrm{d}p,$$

where $z=\frac{a^2-(b-c)^2}{4bc}$, $b-a<c<b+a$, $a=\sqrt{1-p^2}\sinh\frac{t}{2}$, $b=\sinh\frac{x}{2}$, $c=\sinh\frac{x'}{2}$.

Proof. Essentially by the change of variables $y'=(\cosh\frac{t}{2}+p\sinh\frac{t}{2})y$, $-1<p<1$. □

Lemma 3. *If W_μ^ν is a solution of $(a)'$, then we have*:

(i) $A_t^\mu[(\sinh t)^{2\mu}W_{-\mu}^\nu(t,x)]=(\sinh t)^{2\mu}A_t^{-\mu}W_{-\mu}^\nu(t,x)$.
(ii) $(\sinh t)^{2\mu}W_{-\mu}^\nu(t,x)$ *satisfies equation $(a)'$ in $(E_\mu^\nu)'$.*
(iii) $W_\mu^{1-\frac{n}{2}}(t,r)$ *and* $(\sinh t)^{2\mu}W_{-\mu}^{1-\frac{n}{2}}(t,r)$ *satisfies equation (a) where $r=d(x,x')$.*

Proof. (i) By simple computation, (ii) is a consequence of (i), and since the radial part of L_n is $L_{n,r}=A_r^{1-\frac{n}{2}}$ we can see (iii) from (ii). □

Lemma 4. *For $0<t$ and $x,x'\in\mathbb{H}^n$ let $W_{n,\mu}(t,x,x')=C_{n,\mu}(\sinh^2\frac{t}{2}-\sinh^2\frac{r}{2})^{\mu-\frac{n}{2}}$ with $C_{n,\mu}=\frac{4^\mu\Gamma(1+\mu)}{2^n\pi^{\frac{n}{2}}\Gamma(1+\mu-\frac{n}{2})}$ and $r=d(x,x')$, then we have*:

(i) $W_{n,\mu}(t,x,x')=\begin{cases}\alpha_{n,\mu}(\frac{\partial}{\sinh t\partial t})^{\frac{n-1}{2}}(\sinh^2\frac{t}{2}-\sinh^2\frac{r}{2})^{\mu-\frac{1}{2}} & \text{when } n \text{ is odd}\\ \beta_{n,\mu}(\frac{\partial}{\sinh t\partial t})^{\frac{n}{2}}(\sinh^2\frac{t}{2}-\sinh^2\frac{r}{2})^{\mu} & \text{when } n \text{ is even.}\end{cases}$
(ii) $W_{n,\mu}(t,x,x')$ *satisfies the equation (a).*

The proof of this lemma is simple and in consequence is left to the reader.

Lemma 5. *Let J_ν be the Bessel function, then we have*:

(i) $A_x^\nu[\sinh^\nu\frac{x}{2}J_{-\nu}(y\sinh\frac{x}{2})]=\frac{1}{4}\sinh^\nu\frac{x}{2}B_y^\nu[J_{-\nu}(y\sinh\frac{x}{2})]$.
(ii) $A_x^\nu[\sinh^\nu\frac{x}{2}\cosh^{2\nu}\frac{x}{2}J_{-\nu}(y\sinh\frac{x}{2})]=\frac{1}{4}\sinh^\nu\frac{x}{2}\cosh^{2\nu}\frac{x}{2}B_y^{-\nu}[J_{-\nu}(y\sinh\frac{x}{2})]$, *where* $B_y^\nu=y^2\frac{\partial^2}{\partial y^2}+(3-2\nu)y\frac{\partial}{\partial y}+(1-\nu)^2-y^2$.
(iii) $\int(B_y^\nu\phi)\psi\,\mathrm{d}y=\int\phi(C_y^\nu\psi)\,\mathrm{d}y$, *for $\phi\in L^1_{loc}(\mathbb{R}^+)$ and $\psi\in D(\mathbb{R}^+)$ with* $C_y^\nu=y^2\frac{\partial^2}{\partial y^2}+(1+2\nu)y\frac{\partial}{\partial y}+\nu^2-y^2$.
(iv) *The function* $\psi(t,y,y')=y^{-\nu}J_0(\sqrt{4yy'\sinh^2\frac{t}{4}-(y-y')^2})$ *satisfies the equation* $\frac{1}{4}C_y^\nu\psi(t,y,y')=\frac{\partial^2}{\partial t^2}\psi(t,y,y')$.

Proof.

- For (i) and (ii), we transform the derivatives with respect to x to derivatives with respect to y, and we use the Bessel equation:

$$\left[z^2\frac{\partial^2}{\partial z^2}+z\frac{\partial}{\partial z}+z^2-\nu^2\right]J_{-\nu}(z)=0,\quad z=y\sinh\frac{x}{2}.$$

- For (iii), we perform integrations by parts to get the adjoint operator C_y^ν.
- For (iv), using the change of functions $\psi(t,y,y')=y^{-\nu}\phi(t,y,y')$ and the change of variables $z=\sqrt{2yy'\cosh\frac{t}{2}-y^2-y'^2}$, we get the Bessel equation:

$$\left[z^2\frac{\partial^2}{\partial z^2}+z\frac{\partial}{\partial z}+z^2\right]\Phi(z)=0.\qquad\square$$

3. The classical Euler–Poisson–Darboux equation

Proof of Theorem 1.

- To prove that $U(t,x)$ satisfies the equation (a), we use Lemmas 3 and 4.

- To see the initial conditions, we introduce the polar coordinates centralized in x:

$$x' = x + \tan\frac{r}{2}\omega, \quad \omega \in \mathbb{S}^{n-1},$$

and the change of variable $\sinh\frac{r}{2} = (\sinh\frac{t}{2})s,\ 0 < s < 1.$ □

4. The radial wave equation

Proof of Theorem 2.

- To prove that the kernel $W(t, x, x')$ given in (1.1) satisfies the equation $(a)'$, $(\mu = \frac{1}{2})$, we use Lemma 5.
- To see the initial conditions, we use Lemma 2 and the change of variables:

$$x' = 2\sinh^{-1}\left(\frac{\sinh\frac{x}{2} + q\sqrt{1-p^2}\sinh\frac{t}{2}}{\cosh\frac{t}{2} + p\sinh\frac{t}{2}}\right), \quad -1 < q < 1. \qquad \square$$

5. The radial Euler–Poisson–Darboux equation

Proof of Theorem 3.

- To prove that $U(t, x)$ satisfies the equation $(a)'$, we use Lemmas 3, 4 and 5.
- To see the initial conditions, we use Lemma 2 and the change of variables:

$$\sinh\frac{z}{2} = s\sinh\frac{t}{2} \quad \text{and} \quad x' = 2\sinh^{-1}\left(\sinh\frac{x}{2} + q\sqrt{1-p^2 s}\sinh\frac{t}{2}\right), \quad -1 < q < 1. \qquad \square$$

Remark. The explicit formulas in Theorems 1, 2 and 3 are obtained at last formally by using essentially the Jacobi transform in (2.1).

6. Applications

Corollary 1 *(The classical wave equation in the hyperbolic space of dimension n). (See [3], [7] and [1, Proposition 2.2].) We let $\mu \to \frac{1}{2}$ in Theorem 1. We obtain the solution of the Cauchy problem for the classical wave equation in $\mathbb{H}^n$ $(f = 0)$:*

$$U(t, x) = \frac{1}{2(2\pi)^m}\left(\frac{\partial}{\sinh t\,\partial t}\right)^{m-1}\frac{1}{\sinh t}\int_{S_t(x)} g(x')\,\mathrm{d}\mu(x')$$

when n is odd $(n = 2m + 1)$, where $S_t(x)$ is the sphere of radius t around x.

$$U(t, x) = \frac{1}{\sqrt{2}(2\pi)^m}\left(\frac{\partial}{\sinh t\,\partial t}\right)^{m-1}\int_{\mathbb{H}^n} g(x')\Re\frac{1}{\sqrt{\cosh(t) - \cosh(d(x, x'))}}\,\mathrm{d}\mu(x')$$

when n is even $(n = 2m)$, where $d(x, x')$ is the hyperbolic distance between x and x'.

Corollary 2 *(The radial wave equation in the hyperbolic one-dimensional space). We let $\mu \to \frac{1}{2}$ in Theorem 3. We obtain the solution of the Cauchy problem for the radial wave equation (see Theorem 2).*

7. Numerical trials

Example. When $\nu = -\frac{1}{2}$, the radial wave problem

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} + 2\coth x\frac{\partial}{\partial x} + 1\right)U(t, x) = \frac{\partial^2}{\partial t^2}U(t, x) \\ U(0, x) = 0, \quad U_t(0, x) = \sinh x \end{cases} \qquad (P)$$

has a unique solution given by:

$$U(t, x) = \frac{\cosh(2x)\sinh(2t) - 2t}{4\sinh x}.$$

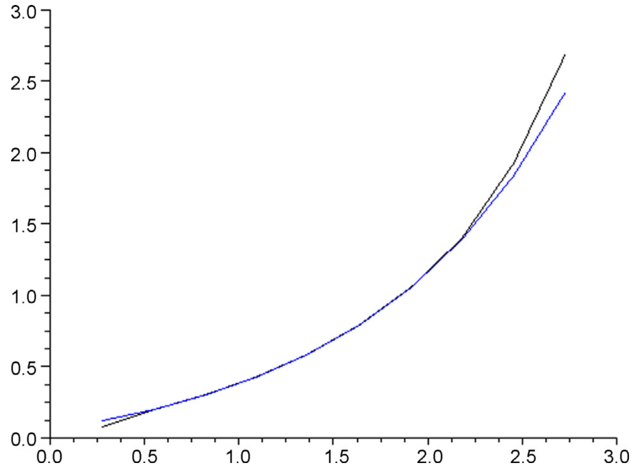

Fig. 1. Representation of the two solutions to the radial wave problem.

We compare the exact solution with the approximate solution obtained by discretization of an interval $[A, B]$, $A > 0$ with a step Δx and a discretization of time with a step Δt (see Fig. 1).

Let $x_j = A + j\Delta x$, $1 \leqslant j \leqslant n_x$, $L = B - A$, $\Delta x = L/(n_x + 1)$ and $t_n = n_t \Delta t$.

Numerically solve the problem (P) means finding a discrete function U defined in points (x_j, t_n), we note U_j^n the values of U at these points. The function U is obtained as the solution of a discrete problem:

$$\begin{cases} \big[1-\theta-(1-\eta)R_j\big]U_{j-1}^{n+1} - \big[2(1-\theta)+r_1\big]U_j^{n+1} + \big[1-\theta+(1-\eta)R_j\big]U_{j+1}^{n+1} \\ \quad = (\eta R_j - \theta)U_{j-1}^n + (2\theta - 2r_1 - r_2)U_j^n - (\theta + \eta R_j)U_{j+1}^n + r_1 U_j^{n-1} \\ U_j^0 = 0, \quad U_j^{-1} = -(\Delta t) g(x_j) \end{cases} \qquad (\widetilde{P})$$

where $r_1 = \frac{(\Delta x)^2}{(\Delta t)^2}$, $r_2 = (\Delta x)^2$ and $R_j = \frac{\Delta x}{\tanh(x_j)}$.

(We take $A = 0.001$, $B = 3$, $n_x = 10$, $n_t = 30$, $\Delta t = 0.01$ and $\theta = \eta = 0.5$.)

References

[1] Ulrich Bunke, Martin Olbrich, The wave kernel for the Laplacian on the classical locally symmetric spaces of rank one, theta functions, trace formulas and the Selberg zeta function, Ann. Glob. Anal. Geom. 12 (1994) 357–405.
[2] C.O.M. El-hafedh, M.V. Ould Moustapha, Problèmes de Cauchy avec des conditions modifiées pour les équations d'Euler–Poisson–Darboux, C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1219–1224.
[3] A. Intissar, M.V. Ould Moustapha, Solution explicite de l'équation des ondes dans l'espace symétrique de type non compact de rang 1, C. R. Acad. Sci. Paris, Ser. I 321 (1995) 77–80.
[4] I.A. Kipriyanov, L.A. Ivanov, Euler–Poisson–Darboux equation on Riemannian space, Sov. Math. Dokl. 260 (4) (1981) 790–794.
[5] I.A. Kipriyanov, L.A. Ivanov, The Cauchy problem for the Euler–Poisson–Darboux equation in a homogeneous symmetric Riemannian space. I, Proc. Steklov Inst. Math. 1 (1987) 159–168.
[6] T.H. Koornwinder, Jacobi functions and analysis on noncompact semisimple Lie groups, in: R.A. Askey, T.H. Koornwinder, W. Schempp (Eds.), Special Functions: Group Theoretical Aspects and Applications, Reidel, Dordrecht, The Netherlands, 1984, pp. 1–85.
[7] P.D. Lax, R.S. Phillips, The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces, J. Funct. Anal. 46 (1982) 280–350.
[8] W. Magnus, F. Oberhettinger, R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, New York, 1966.