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Longitudinal smoothness of the holonomy groupoid

Différentiabilité longitudinale du groupoïde d'holonomie

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ABSTRACT

Iakovos Androulidakis and Georges Skandalis have defined a holonomy groupoid for any singular foliation. This groupoid, whose topology is usually quite bad, is the starting point for the study of longitudinal pseudodifferential calculus on such foliation and its associated index theory. These studies can be highly simplified under the assumption of the holonomy groupoid being longitudinally smooth. In this note, we rephrase the period bounding lemma that asserts that a vector field on a compact manifold admits a strictly positive lower bound for its periodic orbits in order to prove that the holonomy groupoid is always longitudinally smooth.

RÉSUMÉ

Iakovos Androulidakis et Georges Skandalis ont défini un groupoïde d'holonomie pour tout feuilletage singulier. Ce groupoïde, dont la topologie est généralement assez singulière, est le point de départ d'un calcul pseudodifferentiel longitudinal ainsi que d'une théorie de l'indice pour de tels feuilletages. Ces travaux sont grandement simplifiés sous l'hypothèse de différentiabilité longitudinale du groupoïde d'holonomie. Dans cette note, nous réinterprétons le *period bounding lemma* pour montrer que le groupoïde d'holonomie est toujours longitudinalement lisse.

1. Around the period bounding lemma

Recall that the *period bounding lemma* asserts that given a vector field with compact support on a manifold, there is a positive lower bound for the prime periods of periodic orbits (which are not critical points) of the vector field. We will say that the period of a non-periodic curve on a manifold is $+\infty$. Then we have precisely:

Lemma 1.1 *(Period bounding lemma [1,6]). If X is a C^r tangent vector field with compact support on a C^r manifold M with $r \geqslant 2$, there is a real number $\eta > 0$ such that for any x in M, either $X(x) = 0$ or the (prime) period of the integral curve of X passing through x is $\tau_x > \eta$.*

From now on, M is a smooth manifold, $\Gamma_c^\infty(TM)$ is the $C^\infty(M)$-module of compactly supported smooth tangent vector fields on M.

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Let $n \in \mathbb{N}$ and $\mathcal{X} := (X_i)_{i\in[\![1,n]\!]}$ be a family of elements in $\Gamma_c^\infty(TM)$. We denote by $\mathcal{E}$ the submodule of $\Gamma_c^\infty(TM)$ spanned by $\mathcal{X}$. When x belongs to M, we let $I_x := \{f \in C^\infty(M) \mid f(x) = 0\}$ and we consider the quotient linear space $\mathcal{E}_x = \mathcal{E}/I_x\mathcal{E}$. If X belongs to $\mathcal{E}$ we will denote by $[X]_x$ its image in $\mathcal{E}_x$.

For any $\xi = (\xi_i)_{i\in[\![1,n]\!]} \in \mathbb{R}^n$, we define the tangent vector field $X_\xi := \sum_{i=1}^n \xi_i X_i$ and we denote by Ψ_ξ^s its flow at time s. We consider the submersion $r_\mathcal{X} : M \times \mathbb{R}^n \to M;\ \ (x,\xi) \mapsto \Psi_\xi^1(x)$.

Proposition 1.1. *If the family $\mathcal{X}_x := ([X_i]_x)_{i\in[\![1,n]\!]}$ is a basis of $\mathcal{E}_x$, there is a real number $\eta > 0$ such that if $(Id, h) : M \to M \times \mathbb{R}^n$ is a smooth local section of $r_\mathcal{X}$, either $\|h(x)\| \geqslant \eta$ or $h(x) = (x, 0)$.*

Proof. We choose a real number $\lambda > 0$ and we let $B_\lambda := \{t \in \mathbb{R}^n;\ \|t\| < \lambda\}$ be the open ball of radius λ and $S_\lambda := \{t \in \mathbb{R}^n;\ \|t\| = \lambda\}$ the sphere.

Let Z_λ be the smooth tangent vector field on $M \times S_\lambda$ given by $Z_\lambda(y,t) = (X_t(y), 0)$. According to the period bounding lemma, there is a real number $\eta_\lambda > 0$ such that for any $(y,t) \in M \times S_\lambda$, for all $y \in M$ either $Z_\lambda(y,t) = 0$ or the period of the integral curve of Z_λ passing through (y,t) is $\tau_{(y,t)} > \eta_\lambda$. In other words, using that $\Psi_t^s = \Psi_{st}^1$, either $X_t(y) = 0$ or $r_\mathcal{X}(y, st) = \Psi_t^s(y) \neq y$ for any $s \in]0, \eta_\lambda]$.

Choose $\lambda = 1$, let $\eta = \eta_1$ and suppose that $h = (h_i)_{i\in[\![1,n]\!]} : M \to \mathbb{R}^n$ is a smooth map such that $r_\mathcal{X}(y, h(y)) = y$. If $\|h(x)\| < \eta$, the set $V = h^{-1}(B_\eta)$ is an open neighborhood of x on which $X_{h(y)}(y) = 0$. It follows that $X := \sum_{i=1}^n h_i X_i$, which belongs to $\mathcal{E}$, is equal to 0 on V. Looking at the image of X in $\mathcal{E}_x$, we get:

$$[0]_x = [X]_x = \left[\sum_{i=1}^n h_i X_i\right]_x = \sum_{i=1}^n h_i(x)[X_i]_x.$$

Since the family $\mathcal{X}_x$ is a basis of $\mathcal{E}_x$, we deduce that $h_i(x) = 0$ for any $i \in [\![1,n]\!]$. $\square$

2. About the longitudinal smoothness of the holonomy groupoid

As previously, M is a smooth manifold. A *singular foliation* $\mathcal{F}$ on M is a locally finitely generated submodule of $\Gamma_c^\infty(TM)$, stable under the Lie bracket. We first recall briefly the construction of Iakovos Androulidakis and Georges Skandalis [2].

2.1. The holonomy groupoid of a singular foliation [2]

A *bi-submersion* of $\mathcal{F}$ is the data of (N, r_N, s_N) where N is a smooth manifold, $r_N, s_N : N \to M$ are smooth submersions such that[1]:

$$r_N^{-1}(\mathcal{F}) = s_N^{-1}(\mathcal{F}) \quad \text{and} \quad s_N^{-1}(\mathcal{F}) = C_c^\infty(N; \ker ds_N) + C_c^\infty(N; \ker dr_N).$$

The *inverse* of (N, r_N, s_N) is (N, s_N, r_N) and if (T, r_T, s_T) is another bi-submersion for $\mathcal{F}$ the *composition* is given by $(N, r_N, s_N) \circ (T, r_T, s_T) := (N \times_{s_N, r_T} T, r_N \circ p_N, s_T \circ p_T)$, where p_N and p_T are the natural projections respectively of $N \times_{s_N, r_T} T$ on N and on T.

A *morphism* from (N, r_N, s_N) to (T, r_T, s_T) is a smooth map $h : N \to T$ such that $s_T \circ h = s_N$ and $r_T \circ h = r_N$ and it is *local* when it is defined only on an open subset of N.

Finally, a bi-submersion can be *restricted*: if U is an open subset of N, (U, r_U, s_U) is again a bi-submersion, where r_U and s_U are the restriction of r_N and s_N to U.

For x in M, we define the *fiber* of $\mathcal{F}$ at x to be the quotient $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. Let $\mathcal{X} = (X_i)_{i\in[\![1,n]\!]} \in \mathcal{F}^n$ be such that $\mathcal{X}_x = ([X_i]_x)_{i\in[\![1,n]\!]}$ is a basis of $\mathcal{F}_x$. As previously, for any $\xi = (\xi_i)_{i\in[\![1,n]\!]} \in \mathbb{R}^n$, we consider the vector field $X_\xi := \sum_{i=1}^n \xi_i X_i$ and we denote by Ψ_ξ^s its flow at time s. We consider the two smooth submersions from $M \times \mathbb{R}^n$ to M:

$$(s_\mathcal{X}, r_\mathcal{X}) : M \times \mathbb{R}^n \longrightarrow M \times M; \quad (x,\xi) \mapsto \big(x, \Psi_\xi^1(x)\big).$$

Proposition 2.1. *(Cf. Propositions 2.10 and 3.11 of [2].)*

(1) *One can find an open neighborhood W of $(x,0)$ in $M \times \mathbb{R}^n$ such that (W, r_W, s_W) is a bi-submersion, where the map r_W and s_W are the restriction to W of the maps $r_\mathcal{X}$ and $s_\mathcal{X}$ defined above. Such a bi-submersion is called a* path holonomy bi-submersion minimal at *x*.

(2) *Let (N, r_N, s_N) be a bi-submersion, γ in N with $s_N(\gamma) = r_N(\gamma) = x$ and suppose that there exists a smooth local section σ of both r_N and s_N defined on a neighborhood of x in M such that $\sigma(x) = \gamma$. There exists a local morphism of bi-submersions around γ from N to W sending γ on $(x,0)$. Such a morphism is necessarily a submersion at γ.*

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[1] If $h : N \to M$ is a smooth submersion $h^{-1}(\mathcal{F})$ is the vector space generated by tangent vector fields fZ where $f \in C_c^\infty(N)$ and Z is a smooth tangent vector field on N, which is projectable by dh and such that $dh(Z)$ belongs to $\mathcal{F}$.

Notice that any restriction around $(x,0)$ of a path holonomy bi-submersion minimal at x is again a path holonomy bi-submersion minimal at x.

The *path holonomy atlas* $\mathcal{U} = (U_i, r_i, s_i)_{i\in I}$ is the family of bi-submersions of $\mathcal{F}$ generated by the path holonomy bi-submersions and stable under restrictions, compositions and inverses of bi-submersions.

The *holonomy groupoid* of $\mathcal{F}$ is the quotient $G(\mathcal{U}) = \bigsqcup_{i\in I} U_i/\sim$ where $U_i \ni u \sim v \in U_j$ if and only if there is a local morphism from U_i to U_j sending u on v. When (U, r_U, s_U) belongs to $\mathcal{U}$ and $u \in U$, let us denote by $[U, r_U, s_U]_u$ its image in $G(\mathcal{U})$. The structural morphisms of $G(\mathcal{U})$ are given by:

$$\text{source } \mathbf{s}\big([U,r_U,s_U]_u\big) = s_U(u), \qquad \text{range } \boldsymbol{r}\big([U,r_U,s_U]_u\big) = r_U(u), \qquad \text{inverse } [U,r_U,s_U]_u^{-1} = [U,s_U,r_U]_u,$$

$$\text{product } [U,r_U,s_U]_u \cdot [V,r_V,s_V]_v = [U \times_{s_U,r_V} V, r_U \circ p_U, s_V \circ p_V]_{(u,v)} \quad \text{when } s_U(u) = r_V(v),$$

$$\text{units } x \in M \mapsto [W, r_W, s_W]_{(x,0)}, \quad \text{where } W \text{ is any path holonomy bi-submersion minimal at } x.$$

The holonomy groupoid is endowed with the quotient topology, which is quite bad; in particular, the dimension of the fibers may change. We will say that it is *longitudinally smooth* over $x \in M$ when the induced topology on $G(\mathcal{U})_x := \mathbf{s}^{-1}(x)$ makes $G(\mathcal{U})_x$ into a smooth manifold. According to [2], a necessary and sufficient condition for $G(\mathcal{U})_x$ to be smooth is that there is a path holonomy bi-submersion minimal at x, (W, r_W, s_W) such that the restriction of the quotient map $W_x := s_W^{-1}(x) \to G(\mathcal{U})$ is injective. We now show that as a consequence of Proposition 1.1 this is always true.

2.2. Longitudinal smoothness of $G(\mathcal{U})$

The following lemma is a direct consequence of Proposition 2.1.

Lemma 2.1. *For $x \in M$ and $\mathsf{W} = (W, r, s)$ a path holonomy bi-submersion minimal at x, there is a restriction $\widetilde{\mathsf{W}} = (\widetilde{W}, r, s)$ of W around $(x,0)$ and a morphism of bi-submersion $F : \widetilde{\mathsf{W}}^{-1} \circ \widetilde{\mathsf{W}} \to \mathsf{W}$ such that $F(x,0;x,0) = (x,0)$ and the map $T : \widetilde{\mathsf{W}}^{-1} \circ \widetilde{\mathsf{W}} \to \mathsf{W} \times \mathbb{R}^n$, $(y,t;z,\xi) \mapsto (F(y,t,z,\xi), t)$ is injective.*

Proof. Recall that the bi-submersion $\mathsf{W}^{-1} \circ \mathsf{W}$ is $(W \times_r W, s \circ p_1, s \circ p_2)$, where p_i, $i = 1, 2$, are the two canonical projections of $W \times_r W$ on W.

According to Proposition 2.1, one can find a local morphism of bi-submersion $F : \mathsf{W}^{-1} \circ \mathsf{W} \to \mathsf{W}$ such that $F(x,0;x,0) = (x,0)$, which is a submersion. Consider the morphism $j : \mathsf{W} \to \mathsf{W}^{-1} \circ \mathsf{W}$ given by the embedding $(z,\xi) \mapsto (r(z,\xi), 0; z, \xi)$. Then $F \circ j$ is again a local morphism of bi-submersion around $(x,0)$ with values in a bi-submersion minimal at x, thus it remains a submersion at $(x,0)$. So F restricted to $W^0 := \{(y,0;z,\xi) \in W \times_r W\}$ is a submersion at $(x,0;x,0)$. It follows that the map $T : W \times_r W \to W \times \mathbb{R}^n$, $(y,t;z,\xi) \mapsto (F(y,t;z,\xi), t)$ is a local diffeomorphism at $(x,0;x,0)$. $\square$

Proposition 2.2. *The holonomy groupoid of any singular foliation is longitudinally smooth.*

Proof. According to Proposition 1.1 and Lemma 2.1, we can find $\lambda > 0$, an open neighborhood O of x in M and an open neighborhood $\widehat{W}$ of $(x,0)$ in W such that if $\widetilde{W} := O \times B_\lambda$ then $\widetilde{W} \subset \widehat{W}$ and:

(i) if $\sigma = (Id, h) : M \to M \times \mathbb{R}^n$ is a smooth local section of r with values in $\widehat{W}$ then $\sigma(x) = (x,0)$;
(ii) F is defined on $\widetilde{W} \times_r \widetilde{W}$ with values in $\widehat{W}$ and T is injective on $\widetilde{W} \times_r \widetilde{W}$.

Take a local morphism $h : \widetilde{W} \to \widetilde{W}$ and $t \in B_\lambda$. Then $y \in O \mapsto F(y,t;h(y,t))$ is a smooth local section of both s and r with values in $\widehat{W}$. Thus by (i) $F(x,t;h(x,t)) = (x,0)$, whence $T(x,t;h(x,t)) = (x,0;t) = T(x,t;x,t)$. Now, by injectivity of T, we get: $h(x,t) = (x,t)$. In other words, $[\widetilde{W}, r, s]_{(x,t)} = [\widetilde{W}, r, s]_{(x,\xi)}$ if and only if $\xi = t$. $\square$

2.3. The transitive Lie algebroid

For any leaf L of the foliation, $\mathcal{A}_L := \bigcup_{x\in L} \mathcal{F}_x$ inherits from $\mathcal{F}$ a structure of transitive Lie algebroid over L. We let $G(\mathcal{U})|_L$ be the restriction of the groupoid $G(\mathcal{U})$ over L: $G(\mathcal{U})|_L = s^{-1}(L) = r^{-1}(L)$.

Corollary 2.2. *For any leaf L, $G(\mathcal{U})|_L$ is a smooth groupoid that integrates $\mathcal{A}_L$.*

Various consequences of this result may be developed in a forthcoming paper. In particular, it gives information about the Crainic–Fernandes obstruction and equivalently the Mackenzie criteria for integrability of transitive Lie algebroids arising in this way [4,5]. Deeply related to these questions is a recent work of Iakovos Androulidakis and Marco Zambon [3], where they define, for each leaf of a singular foliation, an *essential isotropy group* in order to study the normal form of a singular foliation around a compact leaf. Moreover, they explain how the integrability of A_L (in the sense of Crainic and Fernandes) is related with the discreteness of the essential isotropy group and they show that these essential isotropy groups are discrete if and only if the holonomy groupoid of the foliation is longitudinally smooth.

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