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Mathematical Analysis/Partial Differential Equations

On the asymptotics of a Robin eigenvalue problem

*Asymptotique d'un problème aux valeurs propres avec condition de Robin*Fioralba Cakoni^a, Nicolas Chaulet^b, Housseem Haddar^c^a Department of Mathematical Sciences, University of Delaware, USA^b Department of Mathematics, University College London, UK^c CMAP, École polytechnique, Palaiseau, France

ARTICLE INFO

Article history:

Received 19 June 2013

Accepted 31 July 2013

Available online 20 August 2013

Presented by Alain Bensoussan

ABSTRACT

The considered Robin problem can formally be seen as a small perturbation of a Dirichlet problem. However, due to the sign of the impedance value, its associated eigenvalues converge point-wise to $-\infty$ as the perturbation goes to zero. We prove in this case that Dirichlet eigenpairs are the only accumulation points of the Robin eigenpairs with normalized eigenvectors. We then provide a criterion to select accumulating sequences of eigenvalues and eigenvectors and exhibit their full asymptotic with respect to the small parameter.

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R É S U M É

Le problème de Robin que l'on considère peut formellement être vu comme une petite perturbation d'un problème de Dirichlet. Néanmoins, à cause du signe de l'impédance, ses valeurs propres vont ponctuellement vers $-\infty$ lorsque le petit paramètre tend vers 0. Nous montrons néanmoins que les couples valeurs–vecteurs propres du problème de Dirichlet sont les seuls points d'accumulation des couples valeurs–vecteurs propres de Robin associés à des suites de vecteurs propres normalisés. Nous proposons un critère qui permet de sélectionner les suites de valeurs propres et de vecteurs propres qui s'accumulent sur les valeurs propres et les vecteurs propres de Dirichlet, et nous donnons et justifions leur développement asymptotique complet par rapport au petit paramètre.

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Version française abrégée

Le problème aux valeurs propres (1)–(2) avec le petit paramètre $\delta > 0$ apparaît de manière naturelle dans de nombreuses situations. Il intervient aussi bien dans l'étude d'équations de réaction diffusion (voir [3]) qu'en théorie de la diffraction. En effet, un tel modèle peut être vu comme un modèle approché pour le problème de transmission intérieur associé à la diffraction par un conducteur parfait couvert d'une fine couche de diélectrique (voir [1, chapitre 8]). Il peut être aussi interprété comme un modèle approché pour la diffraction d'une onde électromagnétique par un conducteur parfait, revêtu d'une couche mince de métamatériaux (matériaux pour lesquels les constantes diélectriques sont négatives).

Nous nous concentrons dans cet article à l'étude du comportement asymptotique des valeurs propres associées à (1)–(2) lorsque $\delta \rightarrow 0$. Pour δ fixé, nous savons qu'il existe une suite de valeurs propres réelles $\{\lambda_i^\delta\}_{i=1}^\infty$ ayant $+\infty$ comme seul

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point d'accumulation. De plus, pour δ suffisamment petit, certaines de ces valeurs propres deviennent négatives. Il a été montré (voir [2,4] par exemple) que, pour un domaine Ω de frontière C^1 , pour tout $i \geq 1$ (fixé), $-\delta^2 \lambda_i^\delta \rightarrow 1$ quand $\delta \rightarrow 0$. Nous complétons ce résultat dans deux directions principalement. Tout d'abord, nous montrons (Lemme 2.1) que les valeurs propres de l'opérateur $-\Delta$ dans Ω avec condition de Dirichlet sont les seuls points d'accumulation pour les suites de valeurs propres $\{\lambda^\delta\}_\delta$ associées à des suites de vecteurs propres $\{u^\delta\}_\delta$ ne tendant pas vers 0 en norme L^2 lorsque $\delta \rightarrow 0$. Nous proposons un critère permettant de sélectionner les suites s'accumulant sur les valeurs propres de Dirichlet dans le théorème qui suit.

Théorème 0.1. *Soit une suite $(\lambda^\delta, u^\delta) \in \mathbb{R} \times H^1(\Omega)$ qui satisfait (3) pour tout $\delta > 0$ et qui est telle que $\|u^\delta\|_{H^1(\Omega)} = 1$ et $\lambda^\delta \leq C < +\infty$ pour une constante C indépendante de δ et soit K un ensemble ouvert non vide inclus de manière compacte dans Ω . Alors, la suite (λ^δ) s'accumule sur une valeur propre de Dirichlet si et seulement si il existe $\eta > 0$ et $\delta_0 > 0$ tels que, pour tout $\delta \leq \delta_0$, $\|u^\delta\|_{L^2(K)} \geq \eta$.*

Nous donnons aussi un développement asymptotique à tout ordre de ces suites de valeurs propres. Pour $N \geq 0$, définissons $\Lambda_N^\delta := \sum_{k=0}^N \delta^k \lambda_k$ et $U_N^\delta := \sum_{k=0}^N \delta^k u_k$ deux ansatz pour les valeurs propres et les vecteurs propres de (1)–(2). En remplaçant λ^δ et u^δ par Λ_N^δ et U_N^δ respectivement dans (1)–(2) et en égalisant les termes de même puissance en δ , on est en mesure de calculer tous les λ_i et u_i de manière explicite. Ainsi, on définit λ_0 comme étant une valeur propre du laplacien dans Ω avec condition de Dirichlet associé au vecteur propre u_0 de norme L^2 égale à 1, puis nous définissons $\lambda_1 := \int_\Gamma |\partial u_0 / \partial \nu|^2 ds$, et ainsi de suite. Nous montrons le théorème qui suit en utilisant un résultat de convergence pour le spectre d'un opérateur compact et autoadjoint (voir Lemme 1.1 du chapitre 3 de [5]).

Théorème 0.2. *Pour tout entier N , il existe $C > 0$ et $\delta_0 > 0$ tels que, pour tout $\delta < \delta_0$, $\delta > 0$, il existe une valeur propre $\lambda^\delta > 0$ de (1)–(2) satisfaisant $|\lambda^\delta - \Lambda_N^\delta| \leq C\delta^{N+1}$.*

1. A Robin eigenvalue problem with negative sign

We are interested in the asymptotic behavior of the eigenvalues λ^δ and eigenfunctions $u^\delta \in H^1(\Omega)$ of the following problem:

$$\Delta u^\delta + \lambda^\delta u^\delta = 0 \quad \text{in } \Omega, \quad (1)$$

$$\partial_\nu u^\delta - \frac{1}{\delta} u^\delta = 0 \quad \text{on } \Gamma, \quad (2)$$

with respect to $\delta > 0$ as it approaches 0, where ν is the outward unit normal vector to Γ which is the C^2 -smooth boundary of the bounded connected domain $\Omega \subset \mathbb{R}^d$ for $d \geq 2$.

The eigenvalue problem with Robin boundary condition described by (1)–(2) naturally appears in a number of models related to reaction diffusion problems (see [3]) or scattering theory. For the latter, this eigenvalue problem can be seen as a first approximation to the interior transmission eigenvalue problem associated with the scattering problem by a perfectly conducting body coated with a dielectric layer of width δ (see [1, Chapter 8]). It can also be seen as an approximate model to direct scattering problems for perfect conductors coated with metamaterials.

It is well-known that problem (1)–(2) has an infinite sequence of real eigenvalues $\{\lambda_i^\delta\}_{i=1}^\infty$ accumulating at $+\infty$. However, for sufficiently small δ , some eigenvalues become negative and their number grows to $+\infty$ as $\delta \rightarrow 0$. In fact, for at least C^1 smooth boundary Γ , it is known (see for instance [2,4]) that for every (fixed) $i \geq 1$, $-\delta^2 \lambda_i^\delta \rightarrow 1$ as $\delta \rightarrow 0$.

In Section 2, we complement this result by indicating that Dirichlet eigenvalues for the $-\Delta$ operator in Ω are the only possible finite accumulation points of λ^δ (extending this way the result obtained in [3] for simple geometries), if the associated H^1 normalized eigenfunctions do not L^2 converge to 0 as δ goes to 0. We also prove that eigenvectors associated with other accumulation points concentrate at the boundary (in the sense that they converge to zero in any compact set of Ω). Our main result is given in Section 3; it stipulates that some λ^δ does accumulate at Dirichlet eigenvalues providing a full asymptotic development of these sequences as δ goes to zero.

2. Characterization of Robin eigenvalues accumulating at finite points

We give a criterion that distinguishes between eigenvalues that converge to a Dirichlet eigenvalue from others. We recall that (1)–(2) are equivalent to the following variational formulation:

$$\int_\Omega \nabla u^\delta \nabla v \, dx - \frac{1}{\delta} \int_\Gamma u^\delta v \, ds = \lambda^\delta \int_\Omega u^\delta v \, dx \quad \forall v \in H^1(\Omega). \quad (3)$$

Lemma 2.1. *Assume that a sequence $(\lambda^\delta, u^\delta) \in \mathbb{R} \times H^1(\Omega)$ satisfying (3) is such that $\|u^\delta\|_{H^1(\Omega)} = 1$ and $|\lambda^\delta| \leq C$ for some $C > 0$ independent of the δ . Then one can extract a subsequence δ' of δ such that $\lambda^{\delta'} \rightarrow \Lambda_0$ and $\|u^{\delta'} - U_0\|_{L^2(\Omega)} \rightarrow 0$ as $\delta' \rightarrow 0$, where if $U_0 \neq 0$ then (Λ_0, U_0) is some Dirichlet eigenpair for $-\Delta$ in Ω .*

Proof. Since the sequence λ^δ is bounded and u^δ is also bounded in $H^1(\Omega)$, one can extract a subsequence δ' such that $\lambda^{\delta'}$ converges to some $\Lambda_0 \in \mathbb{R}$ and $u^{\delta'}$ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $U_0 \in H^1(\Omega)$ as δ' goes to 0. From (3) one deduces that $\int_\Gamma |u^{\delta'}|^2 ds \leq C\delta'$ for some $C > 0$ independent of δ' , hence $U_0 = 0$ on Γ . Moreover, if $\Lambda_0 \neq 0$, taking $v \in H_0^1(\Omega)$ in (3) and letting $\delta' \rightarrow 0$ proves that U_0 satisfies $\int_\Omega \nabla U_0 \nabla v dx = \Lambda_0 \int_\Omega U_0 v dx$ which proves, if $U_0 \neq 0$, that (Λ_0, U_0) is a Dirichlet eigenpair. \square

We remark that any point on the real axis is a possible accumulation point for $\{\lambda^\delta\}_\delta$. Actually, for a given $i \in \mathbb{N}$ the sequence $\{\lambda_i^\delta\}_\delta$ goes to $-\infty$ continuously. Therefore, for any $\Lambda \in \mathbb{R}$ one can build a sequence $\{\lambda^{\delta_i}\}_{i \in \mathbb{N}}$ such that $\lambda^{\delta_i} = \Lambda$ for any i .

Theorem 2.2. Consider a sequence $(\lambda^\delta, u^\delta) \in \mathbb{R} \times H^1(\Omega)$ satisfying (3) such that $\|u^\delta\|_{H^1(\Omega)} = 1$ and $\lambda^\delta \leq C < +\infty$ for some constant C independent of δ and let K be a non-empty open set compactly included in Ω . Then the sequence (λ^δ) accumulates at Dirichlet eigenvalues if and only if there exist $\eta > 0$ and $\delta_0 > 0$ such that for all $\delta \leq \delta_0$,

$$\|u^\delta\|_{L^2(K)} \geq \eta. \tag{4}$$

Proof. First, let us assume that (4) holds. Then take $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ in K and choose $v = \psi^2 u^\delta$ in (3). By developing $\nabla(\psi^2 u^\delta)$ and using Young's inequality we obtain:

$$0 \leq \frac{3}{4} \int_\Omega \psi^2 |\nabla u^\delta|^2 dx \leq \lambda^\delta \int_\Omega \psi^2 u_\delta^2 dx + 4 \int_\Omega |\nabla \psi|^2 u_\delta^2 dx.$$

This implies $\lambda^\delta \geq -4 \|\nabla \psi\|_{L^\infty(\Omega)}^2 / \eta$. Then, by Lemma 2.1, we obtain that accumulation points are either 0 or Dirichlet eigenvalues, but zero is excluded by (4).

Conversely, if (λ^δ) accumulates at Dirichlet eigenvalues, the number of these accumulation points is finite. Then, by Lemma 2.1, and since eigenspaces have finite dimensions and $\|u\|_{L^2(K)} > 0$ for all eigenfunctions, accumulation points of $\|u^\delta\|_{L^2(K)}$ are finite discrete positive numbers. This proves (4). \square

Lemma 2.1 and Theorem 2.2 prove in particular that accumulating points for Robin eigenpairs $(\lambda^\delta, u^\delta) \in \mathbb{R} \times H^1(\Omega)$ such that $\|u^\delta\|_{H^1(\Omega)} = 1$ are only Dirichlet eigenpairs.

3. Asymptotic of Robin eigenvalues accumulating at Dirichlet eigenvalues

First, it is easy to check that $(\lambda^\delta, u^\delta)$ is a solution of (1)–(2) if and only if $\mu^\delta = \lambda^\delta + \frac{\alpha}{\delta^2}$ for some positive constant $\alpha > 0$ and $u^\delta \in H^1(\Omega)$ solve:

$$\int_\Omega \nabla u^\delta \nabla v dx - \frac{1}{\delta} \int_\Gamma u^\delta v ds + \frac{\alpha}{\delta^2} \int_\Omega u^\delta v dx = \mu^\delta \int_\Omega u^\delta v dx \quad \text{for all } v \in H^1(\Omega). \tag{5}$$

In the space of $H^1(\Omega)$ -functions let us introduce the δ -dependence norm $\|u\|_{H_\delta^1(\Omega)}^2 := \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{\delta^2} \|u\|_{L^2(\Omega)}^2$. We can prove a coercivity result for the variational formulation (5) in $H_\delta^1(\Omega)$ thanks to the following lemma, which is obtained by using the inequality $\|u\|_{L^2(\Gamma)}^2 \leq C(\|\nabla u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}^2)$.

Lemma 3.1. There exist positive constants α, θ and δ_0 such that for all $\delta \leq \delta_0$:

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{\delta} \int_\Gamma |u|^2 ds + \frac{\alpha}{\delta^2} \int_\Omega |u|^2 dx \geq \theta \left(\int_\Omega |\nabla u|^2 dx + \frac{1}{\delta^2} \int_\Omega |u|^2 dx \right) \quad \forall u \in H^1(\Omega).$$

Note that from Lemma 3.1, we also have that problem (5) can be written as a generalized eigenvalue problem $(A^\delta u, v)_{H_\delta^1(\Omega)} = \mu^\delta (B^\delta u, v)_{H_\delta^1(\Omega)}$, where the bounded linear operators $A^\delta : H_\delta^1(\Omega) \rightarrow H_\delta^1(\Omega)$ and $B^\delta : H_\delta^1(\Omega) \rightarrow H_\delta^1(\Omega)$ are defined by:

$$(A^\delta u, v)_{H_\delta^1(\Omega)} := \int_\Omega \nabla u \nabla v dx - \frac{1}{\delta} \int_\Gamma u v ds + \frac{\alpha}{\delta^2} \int_\Omega u v dx \quad \text{and} \quad (B^\delta u, v)_{H_\delta^1(\Omega)} := \int_\Omega u v dx$$

for all $u, v \in H^1(\Omega)$. The operator $A^\delta : H_\delta^1(\Omega) \rightarrow H_\delta^1(\Omega)$ is self-adjoint, coercive with coercivity constant independent of δ from Lemma 3.1 and it satisfies $\|A^\delta\| \leq C$ with a constant $C > 0$ independent of δ , whereas the operator $B^\delta : H_\delta^1(\Omega) \rightarrow$

$H^1_\delta(\Omega)$ is self-adjoint and compact. Hence, it is known that there exists a sequence of $\mu_k^\delta > 0, k = 0, \dots, +\infty$ accumulating to infinity such that $1/\mu_k^\delta$ are the eigenvalues of the compact self-adjoint operator $(A^\delta)^{-1/2}B^\delta(A^\delta)^{-1/2}$ and the μ_k^δ are eigenvalues of (5).

3.1. Formal asymptotic of the positive eigenvalues

Let us take $(\lambda^\delta, u^\delta)$ a solution to (1)–(2) and introduce the ansatz $U_N^\delta := \sum_{k=0}^N \delta^k u_k$ for u^δ and $\Lambda_N^\delta := \sum_{k=0}^N \delta^k \lambda_k$ for λ^δ . Plugging these two expressions into (1)–(2) enables us to compute all the terms in the expansions explicitly by equating the same powers of δ . To this end, this process first defines λ_0 as being an eigenvalue of $-\Delta$ with Dirichlet boundary conditions in the domain Ω with the corresponding eigenvector u_0 normalized such that $\|u_0\|_{L^2(\Omega)} = 1$. Let us assume that λ_0 is simple; otherwise, the definition of the higher order terms in the expansion of λ^δ is much more involved. Next, for some $k > 0$, let us assume that u_p and λ_p for $p < k$ are known. Then, the function $u_k \in H^1(\Omega)$ must be a solution to:

$$\Delta u_k + \lambda_0 u_k = - \sum_{p=0}^{k-1} \lambda_{k-p} u_p \quad \text{in } \Omega, \quad u_k = \partial_\nu u_{k-1} \quad \text{on } \Gamma \quad \text{and} \quad \int_\Omega u_k u_0 \, dx = 0,$$

where the latter is the compatibility condition that guaranties the existence of u_k . Here, we use the convention that the terms with negative indices are 0. The compatibility condition determines the value of λ_k to $\lambda_k := \int_\Gamma \partial_\nu u_{k-1} \partial_\nu u_0$ and u_k is uniquely defined and for every k there exists $C > 0$ such that $\|u_k\|_{H^2(\Omega)} \leq C \|u_0\|_{H^1(\Omega)}$. In addition, for all $v \in H^1(\Omega)$ and $k > 0$, u_k satisfies the following variational equality:

$$\int_\Omega \nabla u_k \nabla v \, dx = \sum_{p=0}^k \lambda_{k-p} \int_\Omega u_p v \, dx + \int_\Gamma \partial_\nu u_k v \, ds. \tag{6}$$

3.2. A convergence result

For any two functions $u, v \in H^1(\Omega)$, and for $N > 0$, let us denote by:

$$E_N^\delta(u, v) := (A^\delta u, v)_{H^1_\delta(\Omega)} - \hat{\mu}_N^\delta (B^\delta u, v)_{H^1_\delta(\Omega)}$$

with $\hat{\mu}_N^\delta := \Lambda_N^\delta + \alpha/\delta^2$. Using Eq. (6) and the definition of u_0 , we obtain, after some calculations, that for $N \geq 0$,

$$E_N^\delta(U_N^\delta, v) = \delta^N \int_\Gamma \partial_\nu u_N v \, ds + \sum_{p,k=0, p+k>N}^N \delta^{p+k} \lambda_k \int_\Omega u_p v \, ds.$$

Since u_0 is uniformly bounded with respect to δ in $H^2(\Omega)$, by using the fact that $\|u\|_{L^2(\Gamma)}^2 \leq c\delta \|u\|_{H^1_\delta(\Omega)}^2$ and the bounds on the functions u_k , we obtain that for all $N \geq 0$, there exists $C > 0$ such that for all $\delta > 0$ sufficiently small, $E_N^\delta(U_N^\delta, v) \leq C\delta^{N+1/2} \|v\|_{H^1_\delta(\Omega)}$ for all $v \in H^1(\Omega)$. Note that thanks to the normalization $\|u_0\|_{L^2(\Omega)} = 1$, we have that $\|u_0\|_{H^1_\delta(\Omega)} \geq \delta^{-1}$ and for $N \geq 0$ there exists $C > 0$ such that for all $\delta > 0$ sufficiently small, $\|U_N^\delta\|_{H^1_\delta(\Omega)} \geq C\delta^{-1}$. Hence, setting $\hat{U}_N^\delta := U_N^\delta / \|U_N^\delta\|_{H^1_\delta(\Omega)}$ yields:

$$|E_N^\delta(\hat{U}_N^\delta, v)| \leq C\delta^{N+3/2} \|v\|_{H^1_\delta(\Omega)}, \quad \text{for all } v \in H^1(\Omega). \tag{7}$$

Making use of Lemma 1.1 in Chapter 3 of [5], we can prove the following theorem.

Theorem 3.2. *Let $N \geq 0$ and Λ_N^δ be as above. There exist $C > 0$ and $\delta_0 > 0$ such that for all $\delta > 0, \delta < \delta_0$, there exists an eigenvalue $\lambda^\delta > 0$ of (1)–(2) such that $|\lambda^\delta - \Lambda_N^\delta| \leq C\delta^{N-1/2}$.*

Proof. Let us define $T^\delta := (A^\delta)^{-1/2}B^\delta(A^\delta)^{-1/2}$ as an operator from $H^1_\delta(\Omega)$ into itself. Then (7) becomes $\|T^\delta \hat{U}_N^\delta - \hat{U}_N^\delta / \hat{\mu}_N^\delta\| \leq C\delta^{N+3/2} / |\hat{\mu}_N^\delta|$. From Lemma 1.1 in Chapter 3 of [5], we obtain that there exists an eigenvalue μ^δ of problem (5) such that $|1/\hat{\mu}_N^\delta - 1/\mu^\delta| \leq C\delta^{N+3/2} / |\hat{\mu}_N^\delta|$ and, as a consequence, $|\hat{\mu}_N^\delta - \mu^\delta| \leq C\delta^{N+3/2} |\mu^\delta|$. Therefore, there exists $\tilde{C} > 0$ independent of δ such that $|\mu^\delta| \leq \tilde{C} |\hat{\mu}_N^\delta| \leq \tilde{C} (\Lambda_N^\delta + \alpha/\delta^2)$, which yields the desired result. \square

This result is not optimal in terms of the power of δ , but since for all $N \geq 0$ the error writes $\lambda^\delta - \Lambda_N^\delta = \lambda^\delta - \Lambda_{N+2}^\delta + \delta^{N+1}\lambda_{N+1} + \delta^{N+2}\lambda_{N+2}$, we finally obtain the following result.

Corollary 3.3. *Let $N \geq 0$ and Λ_N^δ be as above. There exist $C > 0$ and $\delta_0 > 0$ such that for all $\delta > 0, \delta < \delta_0$, there exists an eigenvalue $\lambda^\delta > 0$ of (1)–(2) such that $|\lambda^\delta - \Lambda_N^\delta| \leq C\delta^{N+1}$.*

Acknowledgement

The authors gratefully acknowledge Leonid Parnovski from University College London for noticing a flaw that helped us correct and clarify the statements of Section 2.

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