Division of the Dickson algebra by the Steinberg unstable module

Division de l’algèbre de Dickson par le module instable de Steinberg

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Abstract

We compute the division of the Dickson algebra by the Steinberg unstable module in the category of unstable modules over the mod-2 Steenrod algebra.

Résumé

On détermine la division de l’algèbre de Dickson par le module instable de Steinberg dans la catégorie des modules instables sur l’algèbre de Steenrod modulo 2.

1. Introduction

We work in the category $\mathcal{U}$ of unstable modules over the mod-2 Steenrod algebra $\mathcal{A}$ [8]. For each $H \in \mathcal{U}$ of finite type, let $(- \otimes H) : \mathcal{U} \to \mathcal{U}$ denote the left adjoint functor of the endofunctor $- \otimes H : \mathcal{U} \to \mathcal{U}$. For $V$ an elementary Abelian 2-group, the famous Lannes’ functor $T_V$ is the division by $H^*V$ [5]. Here and in the sequel, $H^*$ denotes the mod-2 singular cohomology functor. For $V, W$ two elementary Abelian 2-groups, the purpose of this note is to determine $(DW : LV) : \mathcal{U}$ where $DW := H^*W_{Aut(W)}$ is the Dickson algebra [1] and $LV$, to be defined below, is the indecomposable summand of the Steinberg summand $MV$ of $H^*V$ [7]. If $\dim V = k$ then $LV$ is also denoted by $L_k$ and we use the same convention for all other notations admitting an elementary Abelian 2-group as index.

Let us explain the motivation for the determination of $(DW : LV) : \mathcal{U}$. In [2] we study the cohomotopy group of a spectrum, $L'(n), n \in \mathbb{N}$, whose mod-2 cohomology, $L'_n$, is an unstable module that has the following minimal $\mathcal{U}$-injective resolution:

$$0 \to L'_n \to L_n \to L_{n-1} \otimes J(1) \to \cdots \to L_1 \otimes J(2^{n-1} - 1) \to J(2^n - 1) \to 0.$$ 

Here $J(k), k \in \mathbb{N}$, is the Brown–Gitler module which corepresents the functor $M \mapsto \text{Hom}(M^{\mathbb{Z}_2}, \mathbb{F}_2)$ [8]. We have spectral sequences computing the cohomotopy of $L'(n)$ [2]:

$$\text{Ext}^r_{\mathcal{U}}(\bigoplus \Sigma^{-t} \mathbb{Z}/2, L'_n) \Longrightarrow \text{Ext}^{r+s}_{\mathcal{A}}(\Sigma^{-t} \mathbb{Z}/2, L'_n) \Longrightarrow [L'(n), \Sigma^{r+s-t} \mathbb{S}]$$.
Here \( \mathcal{M} \) is the category of \( A \)-modules and \( A \)-linear maps of degree zero and \( \mathbb{D}_t \) the \( s \)-th derived functor of the destabilisation functor \( \mathbb{D} : \mathcal{M} \to \mathcal{U} \) \cite{6} which is left adjoint to the inclusion \( \mathcal{U} \hookrightarrow \mathcal{M} \). In order to compute \( \mathrm{Ext}_{\mathcal{U}}^t(\mathbb{D}_s \Sigma^{-t} \mathbb{Z}/2, L_k \otimes f(2^{n-k} - 1)) \) by adjunction, we need to know the division \( (\mathbb{D}_s \Sigma^{-t} \mathbb{Z}/2 : L_k)_{\mathcal{U}} \). Lannes and Zarati showed in \cite{6} that for \( M \) an unstable module, there is an isomorphism \( \mathbb{D}_s \Sigma^{-t} M \cong \Sigma R_s M \) where \( R_s \) is the Singer functor \cite{6}. In particular, for \( s-t \geq 1, \mathbb{D}_s \Sigma^{-t} \mathbb{Z}/2 \cong \Sigma R_s \Sigma^{v^{-1}} \mathbb{Z}/2 \).

The functor \( R_s \) associates to \( \mathbb{Z}/2 \) the Dickson algebra \( D_s \), and to an unstable module \( M \) a certain submodule of \( D_s \otimes M \). One is led to the determination of \( (R_s \Sigma^{-t} \mathbb{Z}/2 : L_k)_{\mathcal{U}} \cong (D_s : L_k)_{\mathcal{U}} \).

Here is the main result of this note.

**Theorem 1.** There is an isomorphism of unstable modules: \( (R_s \Sigma^{-t} \mathbb{Z}/2 : L_k)_{\mathcal{U}} \cong R_{s-k}(M_k) \).

Lannes and Zarati showed in \cite{6} that there is a natural short exact sequence \( 0 \to R_s \Sigma M \to \Sigma R_s M \to \Sigma \Phi R_{s-1} M \to 0 \) for each unstable module \( M \). By **Theorem 1** and by induction on \( t \in \mathbb{N} \), one gets \( (R_s \Sigma^{-t} \mathbb{Z}/2 : L_k)_{\mathcal{U}} \cong R_{s-k}(\Sigma^t M_k) \). As the functors \( R_s \) and \( R_{s-k} \) are exact and commute with colimits \cite{6}, it follows that \( (R_s A : L_k)_{\mathcal{U}} \cong R_{s-k}(A \otimes M_k) \) if \( A \) is a locally finite unstable module.

**Theorem 1** will be proved in Section 2, basing essentially on two technical lemmas whose proofs will be given in Section 3.

**2. Proof of Theorem 1**

Given an elementary Abelian 2-group \( V \), i.e. a finite \( \mathbb{F}_2 \)-vector space, the semi-group \( \text{End}(V) \) acts naturally on the left of \( V \), and thus on the right of \( V^* \) and \( H^* V \) by transposition. The right action of \( \text{Aut}(V) \) on \( V^* \) and \( H^* V \) can be made into a left action by contragredient duality: \( (gf)(v) = f(g^{-1}v), \ g \in \text{Aut}(V), \ f \in V^* \), \( v \in V \).

In order to calculate \( (D_{\mathcal{U}} : L_k)_{\mathcal{U}} \), we recall that \( (H^* W : H^* V)_{\mathcal{U}} \cong F_2^{V \otimes W} \otimes H^* W \) and this is in fact an \( \text{End}(V) \times \text{End}(W) \)-equivariant isomorphism. This can be obtained by using the commutation of Lannes' functor \( T_V \) with the universal enveloping functor \( U : \mathcal{U} \to \mathcal{K} \) \cite{8}. The isomorphism is adjoint to the following composition:

\[
H^* W \overset{\Delta}{\to} H^* W \otimes H^* W \overset{h \otimes \text{id}}{\to} \left[ H^* V \otimes F_2^{V \otimes W} \right] \otimes H^* W
\]

where \( \Delta \) is the coproduct and \( h \) is adjoint to the natural map:

\[
F_2 \left[ \text{Hom}(V, W) \right] \otimes H^* W \cong \text{Hom}_{\mathcal{U}}(H^* W, H^* V) \otimes H^* W \to H^* V.
\]

Now let \( e_2 \) be a primitive idempotent of \( F_2[\text{End}(V)] \) and \( L_2 \cong (H^* V)e_2 \) the indecomposable direct summand of \( H^* V \) associated with \( e_2 \). Here we use the right action of \( \text{End}(V) \) on \( H^* V \). One gets then:

\[
(H^* W : L_2) \cong (e_2 F_2^{V \otimes W}) \otimes H^* W.
\]

As \( (- : L_2)_{\mathcal{U}} \) commutes with taking invariant (as in the case of \( T_V \) \cite{8}), one gets:

\[
(D_{\mathcal{U}} : L_2)_{\mathcal{U}} \cong \left( e_2 F_2^{V \otimes W} \right) \otimes H^* W)^{\text{Aut}(W)}.
\]

(1)

Here we consider the contragredient left action of \( \text{Aut}(W) \) on \( H^* W \) and on \( F_2^{V \otimes W} \). To rewrite the isomorphism (1) in a practical way, we use the following two simple facts.

**Fact 1.** Let \( G \) be a group and \( M, N \) two left \( F_2[G] \)-modules with \( M \) finite dimensional. Then the linear isomorphism \( M \otimes N \to \text{Hom}(M^*, N) \) given by \( m \otimes n \mapsto [f \mapsto f(m)n], \ m \in M, \ n \in N, \ f \in M^*, \) is \( G \)-equivariant and induces an isomorphism \( (M \otimes N)^G \cong \text{Hom}_{F_2[G]}(M^*, N) \).

Here \( M^* \) denotes the contragredient dual of \( M \) which is defined to be the linear dual space \( M^* \) equipped with the left \( F_2[G] \)-module structure given by \( (gf)(m) = f(g^{-1}m), \ f \in M^*, \ m \in M \).

**Fact 2.** Let \( E \) be a semi-group acting on the right of a finite set \( S \). Then the composition:

\[
F_2[X]e \to F_2[X] \xrightarrow{\otimes [f \mapsto f(x)]} (F_2^X)^* \to (e F_2^X)^*
\]

is an isomorphism of vector spaces for each idempotent \( e \) in \( F_2[E] \).

In our case, there is an isomorphism \( F_2[V \otimes W]e_2 \cong (e_2 F_2^{V \otimes W})^* \), and this is actually an isomorphism of left \( F_2[\text{Aut}(W)] \)-modules. These above facts permit us to rewrite the isomorphism (1) as follows:

\[
(D_{\mathcal{U}} : L_2)_{\mathcal{U}} \cong \text{Hom}_{F_2[\text{Aut}(W)]}(F_2[V \otimes W]e_2, H^* W).
\]

(2)

Here we consider homomorphisms between left \( F_2[\text{Aut}(W)] \)-modules.
We now specify to the division by the Steinberg summand of $H^* V$ [7]. For this let us fix an ordered basis $(v_1, \ldots, v_k)$ of $V$ and thus identify each endomorphism of $V$ with its representing matrix with respect to this basis. The Steinberg idempotent [9] of $F_2[\text{Aut}(V)]$ is given by:

$$e_V := \sum_{S \in \Sigma, B \in B_V} SB,$$

where $B_V$ denotes the Borel subgroup of lower triangular matrices in $\text{Aut}(V)$ and $\Sigma_V$ the symmetric group on $k$ letters considered as the subgroup of monomial matrices in $\text{Aut}(V)$.

Let $M_V$ be the direct summand of $H^* V$ associated with $e_V$. This unstable module can be further decomposed by decomposing the Steinberg idempotent $e_V$ in $F_2[\text{End}(V)]$. Set $\tilde{e}_V := e_V - e_V I_V e_V$ where $I_V$ denotes the diagonal matrix $\text{diag}(1, \ldots, 1, 0)$ in $\text{End}(V)$. Then according to [4, Remark 2.5], $e_V = \tilde{e}_V + e_V I_V e_V$ is a decomposition of $e_V$ into a sum of primitive idempotents in $F_2[\text{End}(V)]$.

Let $L_V$ denote the indecomposable direct summand of $H^* V$ associated with $\tilde{e}_V$. It follows from the isomorphism (2) that:

$$(D_W : L_V)_{\mathcal{U}} \cong \text{Hom}_{F_2[\text{Aut}(V)]}(F_2[\text{Hom}(V, W)]\tilde{e}_V, H^* W).$$

The following technical lemma, which is crucial for the proof of Theorem 1, implies in particular that the division $(D_W : L_V)_{\mathcal{U}}$ is trivial if $\text{dim } V > \text{dim } W$.

**Lemma 2.** Let $M \in \text{Hom}(V, W)$ with $\text{rank}(M) < \text{dim } V$. Then $M \tilde{e}_V = 0$.

We consider now the case where $\text{dim } V \leq \text{dim } W$. By Lemma 2, we have:

$$F_2[\text{Hom}(V, W)]\tilde{e}_V = F_2[\text{Inj}(V, W)]\tilde{e}_V,$$

where $\text{Inj}(V, W) \subset \text{Hom}(V, W)$ is the subset of monomorphisms $V \hookrightarrow W$. Now it is clear that the left $\text{Aut}(W)$-set $\text{Inj}(V, W)$ is transitive. By fixing a monomorphism $\alpha : V \hookrightarrow W$, one has $\text{Inj}(V, W) = \text{Aut}(W)\alpha$. By Lemma 2 and by transitivity of $\text{Inj}(V, W)$, one gets:

$$F_2[\text{Hom}(V, W)]\tilde{e}_V = F_2[\text{Inj}(V, W)]\tilde{e}_V = F_2[\text{Aut}(W)]\alpha \tilde{e}_V.$$

that is, $F_2[\text{Hom}(V, W)]\tilde{e}_V$ is generated by $\alpha \tilde{e}_V$ as a left $F_2[\text{Aut}(W)]$-submodule of $F_2[\text{Hom}(V, W)]$. The isomorphism (3) is then rewritten as follows:

$$(D_W : L_V)_{\mathcal{U}} \cong \text{Hom}_{F_2[\text{Aut}(W)]}(F_2[\text{Aut}(W)]\alpha \tilde{e}_V, H^* W).$$

Let $\text{Ann}(\alpha \tilde{e}_V) := \{ f \in F_2[\text{Aut}(W)] | f \alpha \tilde{e}_V = 0 \}$ denote the annihilator ideal of $\alpha \tilde{e}_V$. In order to describe this ideal, let $G_\alpha = \{ g \in \text{Aut}(W) | g \alpha = \alpha \}$ be the stabiliser subgroup of $\alpha$ and let $e_\alpha \in F_2[\text{Aut}(W)]$ be an idempotent which lifts $e_V \in F_2[\text{Aut}(V)]$ through $\alpha$.

$$\begin{align*}
V & \xrightarrow{e_V} V \\
\alpha & \downarrow \\
W & \xrightarrow{e_\alpha} W,
\end{align*}$$

that is $\alpha e_V = e_\alpha \alpha$.

**Lemma 3.** The left ideal $\text{Ann}(\tilde{e}_V \alpha)$ of $F_2[\text{Aut}(W)]$ is generated by $(1 - e_\alpha)$ and $(1 - g | g \in G_\alpha)$.

Combining the isomorphism (4) with this lemma gives $(D_W : L_V)_{\mathcal{U}} \cong [e_\alpha H^* W] \cap [H^* W G_\alpha]$. But it is shown in [6] that $R_U(H^* V) \cong H^* W G_\alpha$ and $R_U(M) \triangleq [H^* U \otimes M] \cap R_U(N)$ if $N$ is an unstable module and $M$ is a submodule of $N$. It follows that:

$$(D_W : L_V)_{\mathcal{U}} \cong [H^* U \otimes e_V H^* V] \cap [R_U(H^* V)] \cong R_U(e_V H^* V) \cong R_U(M_V).$$

Theorem 1 is proved.
3. Proof of Lemmas 2 and 3

Using the ordered basis \((v_1, \ldots, v_k)\) of \(V\), we identify the group \(\text{Aut}(V)\) with the general linear group \(\text{GL}_k := \text{GL}_k(\mathbb{F}_2)\). Recall that \(\bar{e}_k = e_k - e_{k-1}\), where \(I_k\) is the diagonal \(k \times k\)-matrix \(\text{diag}(1, \ldots, 1, 0)\) and \(e_k\) is the Steinberg idempotent of \(F_2[\text{GL}_k]\) defined by \(e_k = \sum_{B \in \Sigma_k} \mathbb{B}_k\), \(B_k\) denoting the subgroup of lower triangular matrices in \(\text{GL}_k\) and \(\Sigma_k\) the symmetric group on \(k\) letters. We consider the Steinberg idempotent \(e_{k-1}\) of \(F_2[\text{GL}_{k-1}]\) as an element of \(F_2[\text{GL}_k]\) by considering \(\text{GL}_{k-1}\) as the subgroup of automorphisms of \(V\) preserving \(v_k\). It was proved in [3] that \(I_k e_k I_k = e_{k-1} I_k e_{k-1}\) and \(e_{k-1} e_k = e_k\).

**Proof of Lemma 2.** We need to prove that if \(M\) is an \(m \times k\)-matrix of rank less than \(k\), then \(M \bar{e}_k = 0\). Suppose first that the last column of \(M\) is zero. Then \(M \bar{e}_{k-1}\) is a sum of matrices with trivial last column. So \(M I_k = M\) and \((M \bar{e}_{k-1}) I_k = M \bar{e}_{k-1}\). We have then:

\[
M e_k I_k e_k = M I_k e_k I_k e_k \quad \text{(as } M I_k = M) \\
= M e_{k-1} I_k e_{k-1} e_k \quad \text{(as } I_k e_k I_k = e_{k-1} I_k e_{k-1}) \\
= M e_{k-1} e_{k-1} e_k \quad \text{(as } M e_{k-1} I_k = M e_{k-1}) \\
= M e_k \quad \text{(as } e_{k-1} e_k = e_k )
\]

Hence \(M \bar{e}_k = M e_k - M e_{k-1} e_k = 0\).

Now let \(M\) be an arbitrary \(m \times k\)-matrix of rank less than \(k\). One chooses \(g \in \text{GL}_k\) such that the last column of \(N := Mg\) is trivial. So \(M e_k \in NF_2[\text{GL}_k] e_k\). But it is well known from the work of Steinberg [9] that \(F_2[\text{GL}_k] e_k = F_2[B_k] e_k\). Hence \(M \bar{e}_k \in NF_2[B_k] e_k\). Since \(e_k = e_k\), it follows that \(M \bar{e}_k \in NF_2[B_k] e_k\). The space \(NF_2[B_k] e_k\) is trivial because, for each \(B \in B_k\), the last column of \(NB\) is zero, which implies \(NB \bar{e}_k = 0\) as verified above. The lemma is proved. \(\square\)

We prove now Lemma 3. For this we need the following elementary fact.

**Fact 3.** Let \(G\) be a finite group acting on the left of a finite set \(S\). For \(s \in S\), let \(\text{Ann}(s) := \{f \in F_2[G] \mid f s = 0\}\) denote the annihilator ideal of \(s\) and \(G_s := \{g \in G \mid gs = s\}\) the stabiliser subgroup of \(s\). Then \(\text{Ann}(s)\) is the left ideal generated by \((1 - g) \mid g \in G_s\).

**Proof of Lemma 3.** Let \(f \in F_2[\text{Aut}(W)]\) be an element of \(\text{Ann}(\alpha \bar{e}_V)\), that is \(f \alpha \bar{e}_V = \alpha \bar{e}_V = 0\) in \(F_2[\text{End}(V, W)]\). So \(f \alpha \bar{e}_V = f \alpha \bar{e}_V \bar{e}_V \bar{e}_V = 0\). The first term of the left-hand side is a linear combination of monomorphisms in \(\text{Hom}(V, W)\), while the second is a combination of homomorphisms of rank \(\dim V - 1\); so each term vanishes, thus \(f \alpha \bar{e}_V = 0\). But \(\alpha \bar{e}_V = e_0 \alpha\), so \(f \alpha \bar{e}_V = 0\). This means that \(f e_0\) belongs to the annihilator ideal \(\text{Ann}(\alpha) \subset F_2[\text{Aut}(W)]\) of \(\alpha\). Hence \(f \equiv f(1 - e_0) \mod \text{Ann}(\alpha)\). By the above fact, \(\text{Ann}(\alpha)\) is the left ideal of \(F_2[\text{Aut}(W)]\) generated by \((1 - g) \mid g \in G_\alpha\), so \(f\) belongs to the left ideal of \(F_2[\text{Aut}(W)]\) generated by \((1 - e_0)\) and \((1 - g) \mid g \in G_\alpha\).

The reverse inclusion is verified easily: that \((1 - e_0) \alpha \bar{e}_V = 0\) is because \((1 - e_0) \alpha \bar{e}_V = \alpha \bar{e}_V - \alpha \bar{e}_V \bar{e}_V = 0\) and that \((1 - g) \alpha \bar{e}_V = 0\) is because \((1 - g) \alpha \bar{e}_V = (\alpha - g \alpha) \bar{e}_V = (\alpha - \alpha) \bar{e}_V = 0\). The lemma is proved. \(\square\)

**References**