Harmonic Analysis

The John–Nirenberg inequality with sharp constants

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ABSTRACT

We consider the one-dimensional John–Nirenberg inequality:

\[ \left| \{ x \in I_0 : |f(x) - f_{I_0}| > \alpha \} \right| \leq C_1 |I_0| \exp \left( - \frac{C_2}{\|f\|_*} \alpha \right). \]

A. Korenovskii found that the sharp constant \( C_2 \) here is \( \frac{2}{e} \). It is shown in this paper that if \( C_2 = \frac{2}{e} \), then the best possible constant \( C_1 \) is \( \frac{1}{2} e^4/e \).

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1. Introduction

Let \( I_0 \subset \mathbb{R} \) be an interval and let \( f \) be an integrable function on \( I_0 \). Given a measurable set \( E \subset \mathbb{R} \), denote by \( |E| \) its Lebesgue measure. Given a subinterval \( I \subset I_0 \), set \( f_I = \frac{1}{|I|} \int_I f \) and

\[ \Omega(f; I) = \frac{1}{|I|} \int_I |f(x) - f_I| \, dx. \]

We say that \( f \in BMO(I_0) \) if \( \|f\|_* = \sup_{I \subset I_0} \Omega(f; I) < \infty \). The classical John–Nirenberg inequality [1] says that there are \( C_1, C_2 > 0 \) such that for any \( f \in BMO(I_0) \),

\[ \left| \{ x \in I_0 : |f(x) - f_{I_0}| > \alpha \} \right| \leq C_1 |I_0| \exp \left( - \frac{C_2}{\|f\|_*} \alpha \right) \quad (\alpha > 0). \]

A. Korenovskii [4] (see also [5, p. 77]) found the best possible constant \( C_2 \) in this inequality, namely, he showed that \( C_2 = \frac{2}{e} \):

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\[
\left| \{ x \in I_0 : |f(x) - f_{I_0}| > \alpha \} \right| \leq C_1 |I_0| \exp \left( -\frac{2/e}{\|f\|_*} \alpha \right) \quad (\alpha > 0),
\]
and in general the constant \(2/e\) here cannot be increased.

A question about the sharp \(C_1\) in (1.1) remained open. In [4], (1.1) was proved with \(C_1 = e^{1+2/e} = 5.67323\ldots\). The method of the proof in [4] was based on the Riesz sunrise lemma and on the use of non-increasing rearrangements. In this paper, we give a different proof of (1.1), yielding the sharp constant \(C_1 = \frac{1}{2} e^{4/e} = 2.17792\ldots\).

**Theorem 1.1.** Inequality (1.1) holds with \(C_1 = \frac{1}{2} e^{4/e}\), and this constant is the best possible.

We also use as the main tool the Riesz sunrise lemma. But instead of the rearrangement inequalities, we obtain a direct pointwise estimate for any \(BMO\)-function (see Theorem 2.2 below). The proof of this result is inspired (and close in spirit) by a recent decomposition of an arbitrary measurable function in terms of mean oscillations (see [2,6]).

We mention several recent papers [7,8] where sharp constants in some different John–Nirenberg-type estimates were found by means of the Bellman function method.

2. Proof of Theorem 1.1

We shall use the following version of the Riesz sunrise lemma [3].

**Lemma 2.1.** Let \(g\) be an integrable function on some interval \(I_0 \subset \mathbb{R}\), and suppose \(g_{I_0} \leq \alpha\). Then there is at most countable family of pairwise disjoint subintervals \(I_j \subset I_0\) such that \(g_{I_j} = \alpha\), and \(g(x) \leq \alpha\) for almost all \(x \in I_0 \setminus (\bigcup I_j)\).

Observe that the family \(\{I_j\}\) in Lemma 2.1 may be empty if \(g(x) < \alpha\) a.e. on \(I_0\).

**Theorem 2.2.** Let \(f \in BMO(I_0)\), and let \(0 < \gamma < 1\). Then there is at most countable decreasing sequence of measurable sets \(G_k \subset I_0\) such that \(|G_k| \leq \min(2\gamma^k, 1)|I_0|\) and for a.e. \(x \in I_0\),

\[
|f(x) - f_{I_0}| \leq \frac{\|f\|_*}{2\gamma} \sum_{k=0}^{\infty} \chi_{G_k}(x).
\]

**Proof.** Given an interval \(I \subset I_0\), set \(E(I) = \{x \in I : f(x) > f_I\}\). Let us show that there is at most a countable family of pairwise disjoint subintervals \(I_j \subset I_0\) such that \(\sum_j |I_j| \leq \gamma |I_0|\) and for a.e. \(x \in I_0\),

\[
(f - f_{I_0}) \chi_{E(I_0)} \leq \frac{\|f\|_*}{2\gamma} \chi_{E(I_0)} + \sum_j (f - f_{I_j}) \chi_{E(I_j)}.
\]  

We apply Lemma 2.1 with \(g = f - f_{I_0}\) and \(\alpha = \frac{\|f\|_*}{2\gamma}\). One can assume that \(\alpha > 0\) and the family of intervals \(\{I_j\}\) from Lemma 2.1 is non-empty (since otherwise (2.2) holds trivially only with the first term on the right-hand side). Since \(g_{I_j} = \alpha\), we obtain:

\[
\sum_j |I_j| = \frac{1}{\alpha} \int \bigcup_{I_j} (f - f_{I_0}) \, dx \leq \frac{1}{\alpha} \int_{\{x \in I_0 : f(x) > f_{I_0}\}} (f - f_{I_0}) \, dx
\]

\[
= \frac{1}{2\alpha} \Omega(f; I_0) |I_0| \leq \gamma |I_0|.
\]

Since \(g_{I_j} = \alpha\), we have \(f_{I_j} = f_{I_0} + \alpha\), and hence:

\[
f - f_{I_0} = (f - f_{I_0}) \chi_{I_0 \setminus \bigcup I_j} + \alpha \chi_{\bigcup_{I_j} I_j} + \sum_j (f - f_{I_j}) \chi_{I_j}.
\]

This proves (2.2) since \(f - f_{I_0} \leq \alpha\) a.e. on \(I_0 \setminus \bigcup I_j\).

The sum on the right-hand side of (2.2) consists of the terms of the same form as the left-hand side. Therefore, one can proceed iterating (2.2). Denote \(I_j^1 = I_j\), and let \(I_j^k\) be the intervals obtained after the \(k\)-th step of the process. Iterating (2.2) \(m\) times yields:

\[
(f - f_{I_0}) \chi_{E(I_0)} \leq \frac{\|f\|_*}{2\gamma} \sum_{k=0}^{m} \sum_j \chi_{E(I_j^k)}(x) + \sum_{i} (f - f_{I_j^{m+i}}) \chi_{E(I_j^{m+i})}.
\]
(where \( I_j^0 = I_0 \)). If there is \( m \) such that for any \( i \) each term of the second sum is bounded trivially by \( \| f \|_e \chi_{E(I_{j}^0)} \), we stop the process, and we would obtain the finite sum with respect to \( k \). Otherwise, let \( m \to \infty \). Using that

\[
\left| \bigcup_{i} I_{j}^{m+1} \right| \leq \gamma \left| \bigcup_{i} I_{j}^{m} \right| \leq \cdots \leq \gamma^{m+1}|I_0|,
\]

we get that the support of the second term will tend to a null set. Hence, setting \( E_k = \bigcup I_{j}^{k} \), for a.e. \( x \in E(I_0) \) we obtain:

\[
(f - f_{l_0}) \chi_{E(I_0)} \leq \frac{\| f \|_e}{2\gamma} \left( \chi_{E(I_0)}(x) + \sum_{k=1}^{\infty} \chi_{E_k}(x) \right),
\]

(2.3)

Observe that \( E(I_j) = \{ x \in I_j \}: f(x) > f_{l_0} + \alpha \} \subset E(I_0) \). From this and from the above process we easily get that \( E_{k+1} \subset E_k \).

Also, \( E_k \subset \bigcup I_{j}^{k} \), and hence \( |E_k| \leq \gamma^{k}|I_0| \).

Setting now \( F(I) = \{ x \in I : f(x) \leq f_1 \} \), and applying the same argument to \( (f_{l_0} - f) \chi_{F(I)} \), we obtain:

\[
(f_{l_0} - f) \chi_{F(I_0)} \leq \frac{\| f \|_e}{2\gamma} \left( \chi_{F(I_0)}(x) + \sum_{k=1}^{\infty} \chi_{F_k}(x) \right),
\]

(2.4)

where \( F_{k+1} \subset F_k \) and \( |F_k| \leq \gamma^{k}|I_0| \). Also, \( F_k \cap E_k = \emptyset \). Therefore, summing (2.3) and (2.4) and setting \( G_0 = I_0 \) and \( G_k = E_k \cup F_k \), \( k \geq 1 \), we get (2.1). □

**Proof of Theorem 1.1.** Let us show first that the best possible \( C_1 \) in (1.1) satisfies \( C_1 \geq \frac{1}{2} e^{4/\beta} \). It suffices to give an example of \( f \) on \( I_0 \) such that for any \( \varepsilon > 0 \),

\[
\left| \{ x \in I_0 : |f(x) - f_{l_0}| > 2(1 - \varepsilon) \| f \|_e \} \right| = |I_0|/2.
\]

(2.5)

Let \( I_0 = [0, 1] \) and take \( f = \chi_{[0.1/4]} - \chi_{[3/4, 1]} \). Then \( f_{l_0} = 0 \). Hence, (2.5) would follow from \( \| f \|_e = 1/2 \). To show the latter fact, take an arbitrary \( I \subset I_0 \). It is easy to see that computations reduce to the following cases: \( I \) contains only 1/4 and \( I \) contains both 1/4 and 3/4.

Assume that \( I = (a, b) \), \( 1/4 \in I \), and \( b < 3/4 \). Let \( \alpha = \frac{1}{4} - a \) and \( \beta = b - \frac{1}{4} \). Then \( f_I = \alpha/(\alpha + \beta) \) and:

\[
\Omega(f; I) = \frac{2}{\alpha + \beta} \int_{\{ x \in I : f > f_I \}} (f - f_I) = \frac{2\alpha\beta}{(\alpha + \beta)^2} \leq 1/2
\]

with \( \Omega(f; I) = 1/2 \) if \( \alpha = \beta \).

Consider the second case. Let \( I = (a, b) \), \( a < 1/4 \) and \( b > 3/4 \). Let \( \alpha \) be as above and \( \beta = b - \frac{1}{4} \). Then:

\[
\Omega(f; I) = \frac{2}{\alpha + \beta + 1/2} \int_{\{ x \in I : f > f_I \}} (f - f_I) = \frac{4\alpha(4\beta + 1)}{(2\alpha + 2\beta + 1)^2}.
\]

Since

\[
\sup_{0 \leq \alpha, \beta \leq 1/4} \frac{4\alpha(4\beta + 1)}{(2\alpha + 2\beta + 1)^2} = 1/2,
\]

this proves that \( \| f \|_e = 1/2 \). Therefore, \( C_1 \geq \frac{1}{2} e^{4/\beta} \). Let us show now the converse inequality.

Let \( f \in BMO(I_0) \). Setting \( \psi(x) = \sum_{k=0}^{\infty} \chi_{G_k}(x) \), where \( G_k \) are from **Theorem 2.2**, we have:

\[
\left| \{ x \in I_0 : \psi(x) > \alpha \} \right| = \sum_{k=0}^{\infty} |G_k| \chi_{[k,k+1)}(\alpha)
\]

\[
\leq |I_0| \sum_{k=0}^{\infty} \min(1, 2\gamma^k) \chi_{[k,k+1)}(\alpha).
\]

Hence, by (2.1),

\[
\left| \{ x \in I_0 : |f(x) - f_{l_0}| > \alpha \} \right| \leq \left| \{ x \in I_0 : \psi(x) > 2\gamma \alpha \| f \|_e \} \right| \leq |I_0| \sum_{k=0}^{\infty} \min(2\gamma^k, 1) \chi_{[k,k+1)}(2\gamma \alpha \| f \|_e).
\]
This estimate holds for any $0 < \gamma < 1$. Therefore, taking here the infimum over $0 < \gamma < 1$, we obtain:

$$\left| \{ x \in I_0 : |f(x) - f_{I_0}| > \alpha \} \right| \leq \Phi \left( \frac{2e}{\|f\|_\infty} \alpha \right) |I_0|,$$

where

$$\Phi(\xi) = \inf_{0 < \gamma < 1} \sum_{k=0}^{\infty} \min(2\gamma^k,1) X_{[k,k+1]} (\gamma^k \xi).$$

Thus, the theorem would follow from the following estimate:

$$\Phi(\xi) \leq \frac{1}{2} e^{\frac{\xi}{2}} (\xi > 0).$$

(2.6)

It is easy to see that $\Phi(\xi) = 1$ for $0 < \xi \leq 2/e$, and in this case (2.6) holds trivially. Next, $\Phi(\xi) = \frac{2}{e\xi}$ for $2/e \leq \xi \leq 4/e$.

Using that the function $e^\xi/\xi$ is increasing on $(1, \infty)$ and decreasing on $(0, 1)$, we get:

$$\max_{\xi \in [2/e, A/e]} 2e^\xi/\xi = \frac{1}{2} e^{A/e},$$

verifying (2.6) for $2/e \leq \xi \leq 4/e$.

For $\xi \geq 1$ we estimate $\Phi(\xi)$ as follows. Let $\xi \in [m, m + 1)$, $m \in \mathbb{N}$. Taking $\gamma_i = i/e\xi$ for $i = m$ and $i = m + 1$, we get:

$$\Phi(\xi) \leq 2 \min \left( \frac{m}{e\xi}, \frac{m+1}{e\xi} \right)$$

$$= 2 \left( \frac{m}{e\xi} \right)^m X_{[m,m]}(\xi) + \left( \frac{m+1}{e\xi} \right)^m X_{[m,m+1]}(\xi),$$

(2.7)

where $\tilde{m} = \frac{1}{e} (m+1)m^{-1}$. Using the fact that the function $e^\xi/\xi^m$ is increasing on $(m, \infty)$ and decreasing on $(0, m)$, by (2.7) we obtain that for $\xi \in [m, m + 1]$,

$$\Phi(\xi)e^\xi \leq \left( \frac{m}{e\tilde{m}} \right)^m e^m = 2 \left( \frac{e^{1/(1+m)^m}}{(1 + 1/m)^m} \right) \equiv c_m.$$

Let us show now that the sequence $\{c_m\}$ is decreasing. This would finish the proof since $c_1 = 1/2 e^{A/e}$. Let $\eta(x) = (1 + 1/x)^x$ for $x > 0$, and

$$\nu(x) = (e^{\eta(x)/\eta(x)})^{x+1}.$$

Then $c_m = 2\nu(m)$ and hence it suffices to show that $\nu'(x) < 0$ for $x \geq 1$. We have:

$$\nu'(x) = \nu(x) \left( \log \frac{e}{\eta(x)} - \left( 1 \right. \right. \left. - \eta(x)/e \right) \log(1 + 1/x)^{1+x}).$$

Since $\eta(x)(1 + 1/x) > e$, we get $\mu(x) = \frac{\eta(x)}{x - \eta(x)} > x$. From this and from the fact that the function $(1 + 1/x)^{1+x}$ is decreasing, we obtain:

$$\left( \frac{1}{m+1} \right)^{1/m+1} = (1 + 1/\mu(x))^{1+\mu(x)} < (1 + 1/x)^{1+x},$$

which is equivalent to that $\nu'(x) < 0$. $\square$

References


