Differential Geometry

Regularity of the Kähler–Ricci flow

Régularité du flot de Kähler–Ricci

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1. Introduction

Let $M$ be a Fano $n$-manifold and $g_0$ be any Kähler metric with Kähler class $2\pi c_1(M)$. Consider the normalized Kähler–Ricci flow:

$$\frac{\partial g}{\partial t} = g - \text{Ric}(g), \quad g(0) = g_0.$$  \hfill (1)

It was proved in [1] that (1) has a global solution $g(t)$ for $t \geq 0$. The main problem is to understand the limit of $g(t)$ as $t$ tends to $\infty$.

By Perelman's non-collapsing result [12], there exists $\kappa = \kappa(g_0) > 0$ such that:

$$\text{vol}_{g(t)}(B_{g(t)}(x, r)) \geq \kappa r^{2n}, \quad \forall t \geq 0, \quad r \leq 1.$$  \hfill (2)
For any sequence $t_i \to \infty$, by taking a subsequence if necessary, $(M, g(t_i))$ converge to a limiting length space $(\overline{M}, d)$ in the Gromov–Hausdorff topology:

$$(M, g(t_i)) \xrightarrow{d_{GH}} (\overline{M}, d).$$

The question is the regularity of $\overline{M}$. A desirable picture is given in the following folklore conjecture.1

Conjecture 1.1. (See [18], also see [9].) $(M, g(t))$ converges (at least along a subsequence) to a shrinking Kähler–Ricci soliton with mild singularities.

Here, “mild singularities” may be understood in two ways: (i) a singular set of codimension at least 4, and (ii) a singular set of a normal variety. By extending the partial $C^0$-estimate conjecture [19] to the Kähler–Ricci flow, one can show that these two approaches are actually equivalent (see Section 3 or [22]).

As pointed out by the first named author, this conjecture implies the Yau–Tian–Donaldson conjecture. The conjecture states that a Fano manifold $M$ admits a Kähler–Einstein metrics if it is K-stable. Recently, solutions were provided for this conjecture in the case of Fano manifolds ([20], also see [6–8]).

2. Regularity of the Kähler–Ricci flow

Let $g(t)$ be a normalized Kähler–Ricci flow on a Fano manifold $M$ and $(\overline{M}, d)$ be a sequential limit as phrased in (3). The main regularity result is:

Theorem 2.1. (See [22].) Suppose that for some uniform $p > n$ and $\Lambda < \infty$,

$$\int_M |\text{Ric}(g(t))|^p \, dv_{g(t)} \leq \Lambda. \tag{4}$$

Then the limit $\overline{M}$ is smooth outside a closed subset $\mathcal{S}$ of (real) codimension $\geq 4$ and $d$ is induced by a smooth Kähler–Ricci soliton $g_\infty$ on $\overline{M} \setminus \mathcal{S}$. Moreover, $g(t_i)$ converges to $g_\infty$ in the $C^\infty$-topology outside $\mathcal{S}$.2

The proof of the theorem relies on Perelman’s pseudolocality theorem [12] of Ricci flow and a regularity theory for manifolds with $L^p$ bounded Ricci curvature ($p$ bigger than half dimension) and uniformly local volume non-collapsing condition (2). This is a generalization of the regularity theories of Cheeger–Colding [2–4] and Cheeger–Colding–Tian [5]. The proof can be carried out following the lines given in these papers under the framework established by Petersen and Wei [13,14] on the geometry of manifolds with integral bounded Ricci curvature.

We shall show in [22] a uniform $L^4$ bound on the Ricci curvature along the Kähler–Ricci flow on any Fano manifold. The above regularity result implies:

Corollary 2.2. (See [22].) Conjecture 1.1, i.e., the Hamilton–Tian conjecture, holds for dimension $n \leq 3$.

In the case of Del-Pezzo surfaces, Conjecture 1.1 follows from [23] and [10].

3. Partial $C^0$ estimate of the Kähler–Ricci flow

The partial $C^0$ estimate of Kähler–Einstein manifolds plays the key role in Tian’s program to resolve the Yau–Tian–Donaldson conjecture, see [17–20,11], for example. An extension of the partial $C^0$-estimate conjecture [19] to the Kähler–Ricci flow on Fano manifolds in [22] under the regularity assumption of the limit $\overline{M}$.

Let $u(t)$ denote the Ricci potentials of the Kähler–Ricci flow $g(t)$ that satisfy:

$$\text{Ric}(g(t)) + \partial t u(t) = g(t), \quad \int e^{-u(t)} \, dv_{g(t)} = \text{vol}(M).$$

The Hermitian metrics $\tilde{g}(t) = e^{-\frac{1}{2}u(t)} g(t)$ have $\omega(t)$, the Kähler forms of $g(t)$, as their Chern curvature forms. Let $H(l)$ be the induced metric on $K_M^{-1}$, the $l$-th power of the anti-canonical bundle ($l \geq 1$). Let $\nabla$ and $\tilde{\nabla}$ denote the $(1, 0)$ and $(0, 1)$ part of the Levi-Civita connection, respectively. Then, at any time $t$, we have the Bochner-type formula for $\sigma \in H^0(M, K_M^{-1})$:

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1. It is often referred to as the Hamilton–Tian conjecture (see [18]).

2. The convergence with these properties is usually referred to as the convergence in the Cheeger–Gromov topology, see [17] for instance.
\[ \Delta |\nabla \sigma|^2 = |\nabla \nabla \sigma|^2 + |\bar{\nabla} \nabla \sigma|^2 - (n + 2)l - 1)|\nabla \sigma|^2 - \langle \delta \bar{\partial} u(\nabla \sigma, \cdot), \nabla \sigma \rangle \]  
(6)

and the Weitzenböck-type formulas for \( \xi \in C^\infty(M, T^{1,0}M \otimes K_M^{-1}) \):

\[ \Delta_\partial \xi = \bar{\nabla}^a \bar{\nabla} \xi + (l + 1)\xi - \delta \bar{\partial} u(\xi, \cdot), \]  
(7)

\[ \Delta_\bar{\partial} \xi = \bar{\nabla}^a \bar{\nabla} \xi - (n - 1)\xi, \]  
(8)

where \( \Delta_\partial \) is the Hodge Laplacian of \( \delta \). Since the Sobolev constant under the Kähler–Ricci flow is uniformly bounded \([24]\), the Moser iteration gives the gradient estimate to \( \sigma \in H^0(M, K_M^{-1}) \) and \( l^2 \) estimate to solutions \( \delta \bar{\partial} = \xi \in C^\infty(M, T^{1,0}M \otimes K_M^{-1}) \); compare Lemmas 4.1 and 5.4 of \([20]\). Perelman’s C1 estimate to \( u(t) \) \([16]\) will be used in the iteration arguments.

Now, let \( \{s_{t,l,l}\}_{l=1}^{N_t} \) be an orthonormal basis of \( H^0(M, K_M^{-1}) \) with respect to the \( L^2 \) norm defined by \( H(t) \) and Riemannian volume form, and put:

\[ \rho_{t,l}(x) = \sum_{i=1}^{N_t} |s_{t,l,l}|_{i}^2(x), \quad \forall x \in M. \]  
(9)

By using arguments similar to those in \([11]\) or \([20]\), we can prove:

**Theorem 3.1 (Partial C0 estimate).** (See \([22]\).) If \( (M, g(t)) \xrightarrow{dGH} (M_\infty, g_\infty) \) as phrased in Theorem 2.1, then the partial \( C^0 \) estimate:

\[ \inf_{t_j} \inf_{x \in M} \rho_{t,l}(x) > 0 \]  
(10)

holds for a sequence of \( l \to \infty \).

A direct corollary of this is to refine the regularity in Theorem 2.1.

**Theorem 3.2.** (See \([22]\).) Suppose \( (M, g(t)) \xrightarrow{dGH} (M_\infty, g_\infty) \) as above. Then \( M_\infty \) is a normal projective variety and \( S \) is a subvariety of complex codimension at least 2.

Finally, let us indicate how to deduce the Yau–Tian–Donaldson conjecture from the Hamilton–Tian conjecture. Suppose \( M \) is K-stable as defined in \([18]\). Then, under the Kähler–Ricci flow \( g(t) \), we get a shrinking Kähler–Ricci soliton. From this, together with the uniqueness theorem on shrinking solitons, we can conclude that the Lie algebra of holomorphic vector fields on \( M_\infty \) is reductive. Then the K-stability implies the vanishing of Futaki invariant of \( M_\infty \), consequently, the limit \( (M_\infty, g_\infty) \) is Kähler–Einstein. If \( M_\infty \) is not biholomorphic to \( M \), then the eigenspaces of the first eigenvalues of \( -\Delta_{g(t)} + g^{IJ}(t)u(t)\delta_{IJ} \) will converge to a subspace of potential functions on \( M_\infty \) whose complex gradients are nontrivial holomorphic vector fields, cf. \([25]\). These vector fields induce the required degeneration of \( M \) to \( M_\infty \), with vanishing Futaki invariants. This gives a contradiction to the K-stability of \( M \). So we have:

**Theorem 3.3.** (See \([22]\).) Suppose \( M \) is K-stable. If \( (M, g(t)) \xrightarrow{dGH} (M_\infty, g_\infty) \) as phrased in Theorem 2.1, then \( M \) coincides with \( M_\infty \) and admits a Kähler–Einstein \( g_\infty \).

In view of the regularity of low dimensional Kähler–Ricci flow in Section 2 we have:

**Corollary 3.4.** (See \([22]\).) The Yau–Tian–Donaldson conjecture holds for dimension \( n \leq 3 \).

**References**


