## Differential Geometry

# On extrinsic symmetric spaces with zero mean curvature in Minkowski space-time 

# Sur les espaces symétriques extrinsèques à courbure moyenne nulle dans l'espace-temps de Minkowski 

Jong Ryul Kim

Department of Mathematics, Kunsan National University, Kunsan, 573-701, Republic of Korea

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#### Abstract

For an extrinsic symmetric space $M$ in Minkowski space-time, we prove that if $M$ is spacelike with zero mean curvature, then it is totally geodesic and if $M$ is timelike with zero mean curvature, then it is totally geodesic or it is a flat hypersurface. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É


Pour un espace symétrique extrinsèque $M$ dans l'espace-temps de Minkowski, nous prouvons que, si $M$ est de type espace et à courbure moyenne nulle, alors $M$ est totalement géodésique, tandis que, si $M$ est de type temps à courbure moyenne nulle, il s'agit alors d'une sous-variété totalement géodésique ou d'une hypersurface.
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## 1. Introduction

Let $M$ be a non-degenerate submanifold in Minkowski space-time $\mathbb{R}_{1}^{n}$. It is called a spacelike, timelike submanifold if its tangent space $T_{x} M$ is spacelike, timelike for each $x \in M$, respectively. The reflection for the affine isometry $s_{x}: \mathbb{R}_{1}^{n} \rightarrow \mathbb{R}_{1}^{n}$ is defined by:

$$
s_{x}(x)=x,\left.\quad s_{*}\right|_{T_{x} M}=-\mathrm{id},\left.\quad s_{*}\right|_{\left(T_{x} M\right)^{\perp}}=\text { id }
$$

We call $M$ a symmetric submanifold or extrinsic symmetric space if it is invariant under the reflection at each affine normal space $\left(T_{x} M\right)^{\perp}$, that is, $s_{x}(M)=M$ for all $x \in M$. The covariant derivation of the second fundamental form $\alpha$ and the above reflection $s_{X}$ give:

$$
\left(\nabla_{u}^{\perp} \alpha\right)(v, w)=s_{*}\left(\nabla_{u}^{\perp} \alpha\right)(v, w)=\left(\nabla_{s_{*} u}^{\perp} \alpha\right)\left(s_{*} v, s_{*} w\right)=-\left(\nabla_{u}^{\perp} \alpha\right)(v, w)
$$

for all $x \in M$ and $u, v, w \in T_{\chi} M$. Thus an extrinsic symmetric space has the parallel second fundamental form. The converse holds due to Strübing in [8].

[^0]In [3] Ferus showed that if an extrinsic symmetric space $M$ in Euclidean space $\mathbb{R}^{n}$ has zero mean curvature, then $M$ is totally geodesic. Here we consider an extrinsic symmetric space $M$ in $\mathbb{R}_{1}^{n}$ whose mean curvature is zero. And we show the following theorem:

Theorem. Let $M$ be an extrinsic symmetric space in $\mathbb{R}_{1}^{n}$. If $M$ is spacelike with zero mean curvature, then it is totally geodesic. And if $M$ is timelike with zero mean curvature, then it is totally geodesic or it is a flat hypersurface.

## 2. Proof of the theorem

The Lie algebra of an indefinite extrinsic symmetric space is constructed in [6]. We recall some necessary notations in [6] for the proof of the theorem. Let $M \subset V=\mathbb{R}_{1}^{n}$ be an extrinsic symmetric space and $\hat{K}=\left\langle s_{x} ; x \in M\right\rangle \subset O$ (V) be the group generated by all reflection $s_{x}$. A one-parameter subgroup of the group $\hat{K}$ determined by a geodesic $\gamma, t \longmapsto p_{t}(\gamma):=$ $s_{\gamma(t / 2)} \circ s_{\gamma(0)}=s_{\exp (v / 2)} \circ s_{\gamma(0)}$ with $v=\gamma^{\prime}(0)$ is called a transvection. We denote by $K$ the identity component of the Lie group generated by transvections [6, Lemma 3.1] and let $M=K / K_{x}$, where $K_{x}$ is the isotropy group for some fixed $x \in M$. Let $\mathfrak{k}$ be the Lie algebra of the group $K$. Then we have a Cartan decomposition $\mathfrak{k}=\mathfrak{k}_{+}+\mathfrak{k}$ - with respect to the involution $\sigma$ given by the conjugation of the reflection $s_{\chi}$, which satisfies [1,6]:

$$
\begin{equation*}
\left[\mathfrak{k}_{-}, \mathfrak{k}_{-}\right]=\mathfrak{k}_{+} . \tag{1}
\end{equation*}
$$

The infinitesimal transvection $t_{v} \in \mathfrak{k}_{-}$is given by its differential $\left.\mathrm{d} p_{s}(\gamma)\right|_{s=0}$. We identify $\mathfrak{k}_{-}$with the tangent space $T_{x} M$ and define a metric on $\mathfrak{k}_{-}$such that $\left\langle t_{v}, t_{w}\right\rangle_{\mathfrak{k}_{-}}=\langle v, w\rangle$ for any $v, w \in T_{x} M$. The isotropy action on the tangent space is assumed to be effective in order to get the non-degenerate metric on $\mathfrak{k}_{-}$.

Put $\left\langle\mathfrak{k}_{+}, \mathfrak{k}_{-}\right\rangle=0$ and define the metric on $\mathfrak{k}_{+}$as:

$$
\begin{equation*}
\left\langle A,\left[t_{v}, t_{w}\right]\right\rangle_{\mathfrak{k}_{+}}=\left\langle\left[A, t_{v}\right], t_{w}\right\rangle_{\mathfrak{k}_{-}} \tag{2}
\end{equation*}
$$

for all $A \in \mathfrak{k}_{+}$and $v, w \in T_{x} M$. This is well defined by the property of the curvature tensor $R(v, w) u=-\left[\left[t_{v}, t_{w}\right], t_{u}\right]$ of an intrinsically symmetric space $M$ and the effective isotropy action. Then we get an ad(k)-invariant metric on $\mathfrak{k}$. Put $\mathfrak{p}_{-}=T_{x} M$ and $\mathfrak{p}_{+}=T_{x} M^{\perp}$. A Lie algebra $\mathfrak{k}$ is extended to a Lie algebra $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ [2] by defining a skew symmetric product [,] on $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ :

$$
\begin{equation*}
[A, v]=A_{*} v, \quad\langle A,[v, w]\rangle_{\mathfrak{k}}=\left\langle A_{*} v, w\right\rangle \tag{3}
\end{equation*}
$$

where $*$ denotes the linearized action of $A$. Note that there is another Cartan decomposition $\mathfrak{g}=\mathfrak{g}_{+}+\mathfrak{g}_{-}$by putting $\mathfrak{g}_{+}=\mathfrak{k}_{+}+\mathfrak{p}_{+}$and $\mathfrak{g}_{-}=\mathfrak{k}_{-}+\mathfrak{p}_{-}$.

The following bracket relations of an extrinsic symmetric space will be used to calculate the Killing form $B^{\mathfrak{g}}$ :

$$
\begin{equation*}
\left[t_{v}, w\right]=\alpha(v, w), \quad\left[t_{v}, \eta\right]=-S_{\eta} v \tag{4}
\end{equation*}
$$

where $\alpha$ is the second fundamental form and $S$ is the Shape operator [2,4,5]. And since the linear maps $t: T_{\chi} M \rightarrow \mathfrak{k}_{-}$, $v \mapsto t_{v}$ and $S:\left(T_{x} M\right)^{\perp} \rightarrow S\left(T_{x} M\right), \eta \mapsto S_{\eta}$ are equivariant with respect to the action of $\mathfrak{k}_{+}$, we see that:

$$
\begin{equation*}
\left[A, t_{v}\right]=t_{A_{*} v}, \quad\left[A_{*}, S_{\eta}\right]=S_{A_{*} \eta} \tag{5}
\end{equation*}
$$

for all $A \in \mathfrak{k}_{+}[2]$.
The Killing form of a Lie algebra $\mathfrak{g}$ is defined by $B^{\mathfrak{g}}(v, w)=\operatorname{trace}(\operatorname{ad}(v) \circ \operatorname{ad}(w))$ for all $v, w \in \mathfrak{g}$. Let us denote by $B^{U}(v, w)=\sum_{i} \epsilon_{i}\left\langle\operatorname{ad}(v) \operatorname{ad}(w) u_{i}, u_{i}\right\rangle$ the partial trace of a non-degenerate subspace $U$ in $\mathfrak{g}$, where $\left\{u_{i}\right\}$ is an orthonormal basis of $U$ with $\left\langle u_{i}, u_{j}\right\rangle=\epsilon_{i} \delta_{i j}$. The Killing form of a symmetric space is well known. Let $\mathfrak{k}$ be a Lie algebra of a symmetric space with a Cartan decomposition $\mathfrak{k}=\mathfrak{k}_{+}+\mathfrak{k}_{-}$satisfying $\left[\mathfrak{k}_{-}, \mathfrak{k}_{-}\right]=\mathfrak{k}_{+}$. Since $\operatorname{ad}\left(t_{v}\right)$ is skew symmetric for $t_{v} \in \mathfrak{k}_{-}$and maps $\mathfrak{k}_{-}$to $\mathfrak{k}_{+}$and vice versa, we have $B^{\mathfrak{k}_{-}}=B^{\mathfrak{k}_{+}}$on $\mathfrak{k}_{-}$. Hence we get

$$
B^{\mathfrak{k}}\left(t_{v}, t_{w}\right)=2 B^{\mathfrak{k}}-\left(t_{v}, t_{w}\right)
$$

For an orthonormal basis $\left\{t_{e_{i}}\right\}$ of $\mathfrak{k}_{-}$with $\operatorname{sign} \epsilon_{i}=\left\langle t_{e_{i}}, t_{e_{i}}\right\rangle$, the Killing form $B^{\mathfrak{k}}$ on $\mathfrak{k}_{-}$is calculated by:

$$
\begin{aligned}
B^{\mathfrak{k}}\left(t_{v}, t_{w}\right) & =2 \sum \epsilon_{i}\left\langle\operatorname{ad}\left(t_{v}\right) \operatorname{ad}\left(t_{w}\right) t_{e_{i}}, t_{e_{i}}\right\rangle \\
& =2 \sum \epsilon_{i}\left\langle\left[t_{v},\left[t_{w}, t_{e_{i}}\right]\right], t_{e_{i}}\right\rangle=-2 \sum \epsilon_{i}\left\langle\left[\left[t_{w}, t_{e_{i}}\right], t_{v}\right], t_{e_{i}}\right\rangle=2 \sum \epsilon_{i}\left\langle R\left(w, e_{i}\right) v, e_{i}\right\rangle .
\end{aligned}
$$

So we obtain:

$$
\begin{equation*}
B^{\mathfrak{k}}\left(t_{v}, t_{w}\right)=-2 \operatorname{Ric}\left(t_{v}, t_{w}\right) . \tag{6}
\end{equation*}
$$

Let us denote by $H$ the mean curvature of an extrinsic symmetric space $M$ in $\mathbb{R}_{1}^{n}$. The Killing form $B^{\mathfrak{g}}$ of an extrinsic symmetric space is obtained (see [6]):

$$
\begin{equation*}
B^{\mathfrak{g}}(v, w)=-\langle\alpha(v, w), 2 m H\rangle=B^{\mathfrak{g}}\left(t_{v}, t_{w}\right) \tag{7}
\end{equation*}
$$

for all $v, w \in \mathfrak{p}_{-}$and $m=\operatorname{dim} M$.
Remark 1. (See [7].) Let $\mathfrak{z}(\mathfrak{k})$ be the center of a Lie algebra $\mathfrak{k}$. Then:

$$
X \in \mathfrak{z}(\mathfrak{k}) \Leftrightarrow 0=\langle[X, U], V\rangle=\langle X,[U, V]\rangle \quad \text { for all } U, V \in \mathfrak{k},
$$

hence $X \in[\mathfrak{k}, \mathfrak{k}]^{\perp}$. Thus if we assume $\left[\mathfrak{k}_{-}, \mathfrak{k}_{-}\right]=\mathfrak{k}_{+}$, then $X \in\left[\mathfrak{k}_{+}, \mathfrak{k}_{-}\right]^{\perp}$. Suppose that $\mathfrak{k}$ is solvable and indecomposable. Then $X \in \mathfrak{z}(\mathfrak{k})$ must be lightlike, otherwise we get holonomy invariant $\operatorname{ad}\left(\mathfrak{k}_{+}\right) X=0$ and non-degenerate subspace $\mathbb{R} \cdot X$. For the selfadjoint endomorphism Ric: $\mathfrak{k}_{-} \rightarrow \mathfrak{k}_{-}$, we get by (6):

$$
-2\left\langle\operatorname{Ric}\left(t_{v}\right), \mathfrak{z}(\mathfrak{k})^{\perp}\right\rangle=-2 \operatorname{Ric}\left(t_{v}, \mathfrak{z}(\mathfrak{k})^{\perp}\right)=B^{\mathfrak{k}}\left(t_{v}, \mathfrak{z}(\mathfrak{k})^{\perp}\right) \subseteq B^{\mathfrak{k}}\left(t_{v},\left[A, t_{w}\right]\right)=0
$$

for all $t_{v}, t_{w} \in \mathfrak{k}_{-}$and $A \in \mathfrak{k}_{+}$. Hence $\operatorname{Ric}\left(t_{v}\right) \subset\left(\mathfrak{z}(\mathfrak{k})^{\perp}\right)^{\perp}=\mathfrak{z}(\mathfrak{k})$ for all $t_{v} \in \mathfrak{k}_{-}$. Since $\mathfrak{z}(\mathfrak{k})$ are totally isotropic and Ric: $\mathfrak{k}_{-} \rightarrow \mathfrak{k}_{-}$ is selfadjoint, we get:

$$
0=\left\langle\operatorname{Ric}\left(t_{v}\right), \operatorname{Ric}\left(t_{w}\right)\right\rangle=\left\langle\operatorname{Ric}^{2}\left(t_{v}\right), t_{w}\right\rangle
$$

for all $t_{v}, t_{w} \in k_{-}$. Thus we obtain $\operatorname{Ric}^{2}=0$.
We denote by $\mathbb{R}_{\mu}^{n}$ an $n$-dimensional pseudo-Euclidean space whose metric is given by:

$$
\langle v, w\rangle=-v_{1} w_{1}-\cdots-v_{\mu} w_{\mu}+v_{\mu+1} w_{\mu+1}+\cdots+v_{n} w_{n}
$$

with $0 \leqslant \mu \leqslant n-1$.
Lemma 1. Let $M$ be an extrinsic symmetric space in $\mathbb{R}_{\mu}^{n}$. If the mean curvature of $M$ is zero everywhere, then the Lie algebra $\mathfrak{g}$ of $M$ is solvable.

Proof. If the mean curvature $H$ is zero, then $B^{\mathfrak{g}}\left(t_{v}, t_{w}\right)=-B^{\mathfrak{g}}(v, w)=0$ for all $v, w \in \mathfrak{p}_{-}$by (7). It follows from [kek $\left.\mathfrak{k}_{-}\right]=$ $\mathfrak{k}_{+}$(1) and the $\operatorname{ad}(\mathfrak{g})$-invariant Killing form that:

$$
B^{\mathfrak{g}}\left(\mathfrak{k}_{+}, \mathfrak{k}_{+}\right)=B^{\mathfrak{g}}\left(\mathfrak{k}_{+},\left[\mathfrak{k}_{-}, \mathfrak{k}_{-}\right]\right)=B^{\mathfrak{g}}\left(\left[\mathfrak{k}_{+}, \mathfrak{k}_{-}\right], \mathfrak{k}_{-}\right) \subseteq B^{\mathfrak{g}}\left(\mathfrak{k}_{-}, \mathfrak{k}_{-}\right)=0
$$

and $B^{\mathfrak{g}}\left(\mathfrak{p}_{+}, \mathfrak{p}_{+}\right)=0$ since:

$$
\begin{aligned}
& B^{\mathfrak{g}}\left(\mathfrak{p}_{+},\left[\mathfrak{k}_{+}, \mathfrak{p}_{+}\right]\right)=B^{\mathfrak{g}}\left(\mathfrak{k}_{+},\left[\mathfrak{p}_{+}, \mathfrak{p}_{+}\right]\right) \subseteq B^{\mathfrak{g}}\left(\mathfrak{k}_{+}, \mathfrak{k}_{+}\right)=0 \\
& B^{\mathfrak{g}}\left(\left[\mathfrak{k}_{-}, \mathfrak{p}_{-}\right], \mathfrak{p}_{+}\right)=B^{\mathfrak{g}}\left(\mathfrak{k}_{-},\left[\mathfrak{p}_{-}, \mathfrak{p}_{+}\right]\right) \subseteq B^{\mathfrak{g}}\left(\mathfrak{k}_{-}, \mathfrak{k}_{-}\right)=0
\end{aligned}
$$

Therefore $B^{\mathfrak{g}}$ is solvable by the Cartan Criterion for solvability, that is, $B^{\mathfrak{g}}(X, Y)=0$ for all $X \in \mathfrak{g}$ and $Y \in[\mathfrak{g}, \mathfrak{g}]$.
To show Lemma 2, the following bracket relations in [2] of an extrinsic symmetric space are used:

$$
\begin{equation*}
[v, w]=\left[t_{v}, t_{w}\right], \quad[v, \eta]=t_{S_{\eta} v}, \quad[\eta, \xi] v=-\left[S_{\eta}, S_{\xi}\right] v \tag{8}
\end{equation*}
$$

for any $v, w \in \mathfrak{p}_{-}$and $\eta, \xi \in \mathfrak{p}_{+}$.
Lemma 2. If $X+t_{Y} \in \mathfrak{g}_{-} \cap \mathfrak{z}(\mathfrak{g})$ for $t_{Y} \in \mathfrak{k}_{-}$and $X \in \mathfrak{p}_{-}$, then $\operatorname{Ric}(X, v)=\operatorname{Ric}(Y, v)=0, A X=A Y=0$ and $\alpha(X, v)=\alpha(Y, v)=0$ for all $A \in \mathfrak{k}_{+}$and $v \in \mathfrak{p}_{-}$.

Proof. Let $X+t_{Y} \in \mathfrak{g}_{-} \cap \mathfrak{z}(\mathfrak{g})$ for $t_{Y} \in \mathfrak{k}_{-}$and $X \in \mathfrak{p}_{-}$. Then by the relations (3), (4), (5) and (8), we get:

$$
\begin{aligned}
& {[A, X]=0, \quad\left[t_{w}, X\right]=\alpha(w, X)=0, \quad[w, X]=0, \quad[\eta, X]=t_{S_{\eta} X}=0,} \\
& {\left[A, t_{Y}\right]=t_{A_{*} Y}=0, \quad\left[t_{w}, t_{Y}\right]=[w, Y]=0, \quad\left[w, t_{Y}\right]=-\alpha(w, Y)=0, \quad\left[\eta, t_{Y}\right]=S_{\eta} Y=0}
\end{aligned}
$$

for all $w \in \mathfrak{p}_{-}$and $\eta \in \mathfrak{p}_{+}$. Equivalently we get

$$
\begin{aligned}
& {\left[A, t_{X}\right]=t_{A_{*} X}=0, \quad\left[t_{w}, t_{X}\right]=[w, X]=0} \\
& {\left[w, t_{X}\right]=-\alpha(w, X)=0, \quad\left[\eta, t_{X}\right]=S_{\eta} X=0}
\end{aligned}
$$

and

$$
[A, Y]=0, \quad\left[t_{w}, Y\right]=\alpha(w, Y)=0, \quad[w, Y]=0, \quad[\eta, Y]=t_{S_{\eta} Y}=0
$$

The curvature tensor of a symmetric space shows $\operatorname{Ric}(X, v)=\operatorname{Ric}(Y, v)=0$.

Proof of the theorem. The Killing form $B^{\mathfrak{g}}$ of an extrinsic symmetric space shows that if the mean curvature of an extrinsic symmetric space in $\mathbb{R}_{1}^{n}$ is zero, then $\mathfrak{g}$ is solvable by Lemma 1 . A solvable Lie algebra has the decreasing sequence called commutator series whose end is Abelian ideal. Let us denote by $\mathfrak{z}(\mathfrak{g})$ the center of a Lie algebra $\mathfrak{g}$. If $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{p}+$ is not empty, then we have $S_{\eta} v=0$ by the bracket relation (4) for all $v \in \mathfrak{p}_{-}$and $\eta \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{p}_{+}$. It suffices to consider the quotient Lie algebra $\mathfrak{g} /\left(\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{p}_{+}\right)$which we denote again by $\mathfrak{g}$. So the elements of the center belong to $\mathfrak{g}_{-}$by Remark 1 and Lemma 2 . For a solvable Lie algebra $\mathfrak{g}$, we get $\mathfrak{g}=0$ in the case of $[\mathfrak{g}, \mathfrak{g}]=\mathfrak{g}$. So $M$ is trivially totally geodesic by (4). Thus we need to look at the nontrivial case of $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$. First, if $M$ is spacelike, then there is a tangent vector $X \in \mathfrak{z}(\mathfrak{g})$ satisfying $\alpha(X, v)=0$ for all $v \in \mathfrak{p}_{-}$by Lemma 2. The quotient Lie algebra $\mathfrak{g} / \mathfrak{z}(\mathfrak{g})$ is also solvable. By the induction arguments, we see that $M$ is totally geodesic. Second, in the case of timelike $M$ with $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$, we may have a lightlike tangent vector $X \in \mathfrak{z}(\mathfrak{g})$ with $\alpha(X, v)=0$ for all $v \in \mathfrak{p}_{-}$by Lemma 2. Otherwise $M$ is totally geodesic by the first induction arguments. So after factoring out (if necessary) a totally geodesic Riemannian extrinsic space by the first arguments, we get an indecomposable Lie algebra. Thus we need to look at the case where $\operatorname{Ric}^{2}=0$ by Remark 1 . Hence we consider $\operatorname{Ric}(X, v)=0$ for a lightlike $X \in \mathfrak{p}_{-} \cap \mathfrak{z}(\mathfrak{g})$ and all $v \in \mathfrak{p}_{-}$by Lemma 2 and an indecomposable Lie algebra $\mathfrak{g}$. Given the above $X$, we take an orthonormal basis $\left\{e_{i}\right\}_{i=1}^{m}$ of $\mathfrak{p}_{-}$such that $X=e_{1}+e_{m}$ and $N=-\frac{1}{2} e_{1}+\frac{1}{2} e_{m}$ for timelike $e_{1}$ and spacelike $e_{m}$. Since the Ricci tensor is two-step nilpotent with $\operatorname{Ric}(X, v)=0$ for all $v \in \mathfrak{p}_{-}$, we can put:

$$
\begin{equation*}
\operatorname{Ric} X=0, \quad \operatorname{Ric} N=X, \quad \operatorname{Ric}(v, w)=0, \quad \text { for all } v, w \in \operatorname{Span}\{X, N\}^{\perp} \tag{9}
\end{equation*}
$$

where $\operatorname{Span}\{X, N\}^{\perp}$ is the orthogonal complement of $\operatorname{Span}\{X, N\}$. Thus it follows from:

$$
\begin{aligned}
& \operatorname{Ric}\left(X, e_{1}\right)=\operatorname{Ric}\left(e_{1}+e_{m}, e_{1}\right)=\operatorname{Ric}\left(e_{1}, e_{1}\right)+\operatorname{Ric}\left(e_{1}, e_{m}\right)=0 \\
& \operatorname{Ric}\left(X, e_{m}\right)=\operatorname{Ric}\left(e_{1}+e_{m}, e_{m}\right)=\operatorname{Ric}\left(e_{1}, e_{m}\right)+\operatorname{Ric}\left(e_{m}, e_{m}\right)=0
\end{aligned}
$$

that:

$$
\begin{equation*}
\operatorname{Ric}\left(e_{1}, e_{1}\right)=\operatorname{Ric}\left(e_{m}, e_{m}\right)=-\operatorname{Ric}\left(e_{1}, e_{m}\right) \tag{10}
\end{equation*}
$$

Similarly from $\alpha\left(X, e_{1}\right)=0$ and $\alpha\left(X, e_{m}\right)=0$, we get:

$$
\begin{equation*}
\alpha\left(e_{1}, e_{1}\right)=\alpha\left(e_{m}, e_{m}\right)=-\alpha\left(e_{1}, e_{m}\right) \tag{11}
\end{equation*}
$$

The Ricci equation:

$$
\operatorname{Ric}(v, w)=\sum \epsilon_{i}\left\langle R\left(v, e_{i}\right) e_{i}, w\right\rangle=\langle\alpha(v, w), m H\rangle-\sum \epsilon_{i}\left\langle\alpha\left(v, e_{i}\right), \alpha\left(e_{i}, w\right)\right\rangle
$$

shows that if $H=0$, then we have by (11):

$$
\begin{align*}
-\operatorname{Ric}\left(e_{1}, e_{m}\right) & =-\left\langle\alpha\left(e_{1}, e_{1}\right), \alpha\left(e_{1}, e_{m}\right)\right\rangle+\left\langle\alpha\left(e_{1}, e_{m}\right), \alpha\left(e_{m}, e_{m}\right)\right\rangle+\sum_{i \neq 1, m}\left\langle\alpha\left(e_{1}, e_{i}\right), \alpha\left(e_{i}, e_{m}\right)\right\rangle \\
& =\sum_{i \neq 1, m}\left\langle\alpha\left(e_{1}, e_{i}\right), \alpha\left(e_{i}, e_{m}\right)\right\rangle \geqslant 0 \tag{12}
\end{align*}
$$

since $\mathfrak{p}_{+}$is positive definite. In the same way, we have:

$$
\begin{equation*}
-\operatorname{Ric}\left(e_{1}, e_{1}\right)=\sum_{i \neq 1, m}\left\langle\alpha\left(e_{1}, e_{i}\right), \alpha\left(e_{i}, e_{1}\right)\right\rangle \geqslant 0 \tag{13}
\end{equation*}
$$

Therefore it follows from (10), (12) and (13) that:

$$
\begin{equation*}
\operatorname{Ric}\left(e_{1}, e_{1}\right)=\operatorname{Ric}\left(e_{m}, e_{m}\right)=-\operatorname{Ric}\left(e_{1}, e_{m}\right)=0 \tag{14}
\end{equation*}
$$

which leads to the flat Ricci tensor Ric $=0$ together with (9).
Then by (13) and $\alpha\left(X, e_{i}\right)=0$, we get:

$$
\begin{equation*}
\alpha\left(e_{1}, e_{i}\right)=0, \quad \alpha\left(e_{m}, e_{i}\right)=0 \quad(i \neq 1, m) \tag{15}
\end{equation*}
$$

Again the Ricci equation and (15):

$$
\begin{aligned}
0 & =\operatorname{Ric}\left(e_{i}, e_{i}\right)=\left\langle\alpha\left(e_{i}, e_{i}\right), m H\right\rangle-\sum \epsilon_{k}\left\langle\alpha\left(e_{i}, e_{k}\right), \alpha\left(e_{k}, e_{i}\right)\right\rangle \\
& =-\left\langle\alpha\left(e_{i}, e_{1}\right), \alpha\left(e_{1}, e_{i}\right)\right\rangle+\left\langle\alpha\left(e_{i}, e_{m}\right), \alpha\left(e_{m}, e_{i}\right)\right\rangle+\sum_{k \neq 1, m}\left\langle\alpha\left(e_{i}, e_{k}\right), \alpha\left(e_{k}, e_{i}\right)\right\rangle \\
& =\sum_{k \neq 1, m}\left\langle\alpha\left(e_{i}, e_{k}\right), \alpha\left(e_{k}, e_{i}\right)\right\rangle
\end{aligned}
$$

imply:

$$
\begin{equation*}
\alpha\left(e_{i}, e_{k}\right)=0 \quad(i, k \neq 1, m) \tag{16}
\end{equation*}
$$

By the second fundamental form (11), (15), (16) and the Gauss equation, we get:

$$
\begin{aligned}
& R\left(e_{1}, e_{i}, e_{i}, e_{1}\right)=\left\langle\alpha\left(e_{i}, e_{i}\right), \alpha\left(e_{1}, e_{1}\right)\right\rangle-\left\langle\alpha\left(e_{1}, e_{i}\right), \alpha\left(e_{1}, e_{i}\right)\right\rangle=0 \\
& R\left(e_{m}, e_{i}, e_{i}, e_{m}\right)=0, \quad R\left(e_{j}, e_{i}, e_{i}, e_{j}\right)=0 \quad(i, j \neq 1, m)
\end{aligned}
$$

Hence $M$ is flat with one nonzero normal vector $\alpha\left(e_{1}, e_{1}\right)(11)$.

With the same arguments in the proof of theorem, we get:
Corollary. Let $M$ be an extrinsic symmetric space in pseudo-Euclidean space $\mathbb{R}_{\mu}^{n}$. If $M$ is spacelike with zero mean curvature, then $M$ is totally geodesic.

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[^0]:    E-mail address: kimjr0@yahoo.com.
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