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# Algebra Generating regular elements

## Engendrer des éléments réguliers

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#### ABSTRACT

Let *R* be a prime right Goldie ring. A useful fact is that, if  $a, b \in R$  are such that aR + bR contains a regular element, then there exists  $\lambda \in R$  such that  $a + b\lambda$  is regular. We show that the analogous result holds for  $n \ge 1$  pairs of elements: if *R* contains a field of cardinality at least n + 1, and if  $a_i, b_i \in R$  are such that  $a_iR + b_iR$  contains a regular element for  $1 \le i \le n$ , then there exists a single element  $\lambda \in R$  such that  $a_i + b_i\lambda$  is regular for each *i*.

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#### RÉSUMÉ

Soit *R* un anneau de Goldie premier. Un résultat utile est que si  $a, b \in R$  sont tels que, aR + bR contienne un élément régulier, alors il existe  $\lambda \in R$  tel que  $a + b\lambda$  est régulier. Nous montrons qu'un résultat analogue est vrai pour  $n \ge 1$  paires de tels élément : si *R* contient un corps de cardinal > n et si les  $a_i, b_i \in R$  sont tels que  $a_iR + b_iR$  contienne un élément régulier, alors il existe  $\lambda \in R$  tel que  $a_i + b_i\lambda$  est régulier pour tout *i*. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Let *R* be either a prime right Goldie ring or a noetherian ring. A useful fact is that, if  $a, b \in R$  are such that aR + bR contains a regular element, then some  $a + b\lambda$  is regular (see [7, Lemma 1.1], respectively [6, Corollary 2.5]). Recently, Carpentier, De Sole and Kac raised the question of whether an analogous result holds for two or more pairs of elements. This was needed specifically for their paper [2] and is related to their work on differential operators and Poisson structures [1,3,4]. The aim of this note is to prove just such a result (see Theorems 1.2 and 1.4).

It is easy to see that this sort of result fails without some condition on R (see Remark 1.3), and so we need the following hypothesis. We write  $C_R(I)$  or just C(I) for the set of elements of a ring R that become regular modulo an ideal I.

**Definition 1.1.** Let  $n \in \mathbb{N}$ . Then R satisfies  $(*_n)$  if there exist regular central elements  $\lambda_1, \ldots, \lambda_n$  such that  $\lambda_j - \lambda_i$  is regular for all  $1 \leq i < j \leq n$ .

Clearly  $(*_n)$  holds if *R* contains a central subfield *k* of cardinality |k| > n. Similarly, when *R* is prime,  $(*_n)$  holds provided the centre of *R* has cardinality |Z(R)| > n.

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**Theorem 1.2.** Fix an integer n > 0 and let R be a noetherian ring that satisfies  $(*_n)$ . Let  $\{a_i, b_i: 1 \le i \le n\} \subset R$  be such that

 $(a_i R + b_i R) \cap C_R(0) \neq \emptyset$  for  $1 \leq i \leq n$ .

Then there exists  $e \in R$  such that

 $a_i + b_i e \in C_R(0)$  for  $1 \leq i \leq n$ .

**Remark 1.3.** Some condition like  $(*_n)$  is necessary for the theorem to hold. For example, the theorem fails for  $R = \mathbb{Z}/n\mathbb{Z}$ with  $a_i = [-i + n\mathbb{Z}]$  and  $b_i = 1$  for  $1 \le i \le n$ . Indeed, the result fails if *R* even has  $\mathbb{Z}/n\mathbb{Z}$  as a ring-theoretic summand.

If one only wants  $a_1 + b_1 e \in C_R(0)$  to hold then the theorem is proved in [6, Corollary 2.5]. To prove Theorem 1.2 we will apply similar techniques. The main case is when the ring is prime, in which case we will get the following slightly stronger result.

**Theorem 1.4.** Fix an integer n > 0 and let S be a prime right Goldie ring that satisfies  $(*_n)$ . Let I be a non-zero ideal of S (possibly I = S).

Let  $\{a_i, b_i: 1 \leq i \leq n\} \subset R$  be such that

 $(a_i S + b_i S) \cap C_S(0) \neq \emptyset$  for  $1 \leq i \leq n$ . (1.5)

Then there exists  $e \in I$  such that

 $a_i + b_i e \in C_{\mathcal{S}}(0)$  for  $1 \leq i \leq n$ . (1.6)

#### 2. The proofs

We begin by recalling various definitions and results, from [5, Chapter 2]. Let S be a prime right Goldie ring with right Goldie quotient ring Q = Q(S). A right S-module M is uniform if  $M \neq 0$  and every non-zero submodule  $N \subseteq M$ is essential in M. The uniform dimension, udim(M) is the maximum integer n such that M contains a direct sum of n non-zero submodules (or udim $(M) = \infty$  if no bound exists). If  $M \subseteq Q$  then udim(M) is the length of the Q-module  $MQ \cong M \otimes_S Q$ . Finally, for  $a \in S$ ,  $udim(aS) = udim(S) \iff a$  is right regular  $\iff a$  is regular. Given  $m \in M$ , write  $r-ann(m) = \{r \in R: mr = 0\}$  for the right annihilator of m.

The following result expands upon [7, Lemma 1.1].

Lemma 2.1. Let S be a prime right Goldie ring.

- (1) Let  $u, v \in S$  with udim(uS + vS) > udim(uS). Set J = r-ann(u). Then there exists  $x \in S$  such that  $0 \neq vxS$  is uniform,  $vxS \cap$ uS = 0 and  $vx I \neq 0$ ;
- (2) Suppose that  $f, g \in S$  are such that gS is uniform, with  $fS \cap gS = 0$  and  $g \cdot r-ann(f) \neq 0$ . (This holds, in particular, if f = u and g = vx in the notation of part (1).) Then udim(f + g)S > udim(fS).

**Proof.** (1) As *u* cannot be right regular,  $J \neq 0$ . Next, *vS* must contain a cyclic, uniform right ideal L = vyS with  $L \cap uS = \emptyset$ . Since S is prime,  $vyS \neq 0$  and so we can pick  $z \in S$  such that  $vyZ \neq 0$ . Now (1) holds with x = yZ. (2) Use the final 6 lines from the proof of [7, Lemma 1.1].  $\Box$ 

The following more technical lemma forms the heart of the proof of Theorem 1.4.

**Lemma 2.2.** Fix an integer  $n \ge 1$  and let S be a prime right Goldie ring for which  $(*_n)$  holds. Let  $a_1, \ldots, a_n, z_1, \ldots, z_{n-1}, y \in S$  be such that

(a)  $a_i \in C_S(0)$  for  $1 \leq i \leq n - 1$ , while

(b) *yS* is uniform with  $udim(a_nS + yS) > udim(a_nS)$ .

Then there exists  $\lambda \in S$  such that

(1)  $a_i + z_i \lambda \in C_{\mathcal{S}}(0)$  for  $1 \leq i \leq n - 1$  while (2)  $\operatorname{udim}(a_n + y\lambda)S > \operatorname{udim}(a_nS)$ .

**Proof.** We assume by induction that the result is true when *n* is replaced by n - 1 (with the case n = 1 being Lemma 2.1). As S is prime there exists  $s \in S$  such that  $ysy \neq 0$ . Since ysyS is then essential in yS, it follows that  $(a_nS + ysyS)$  is essential in  $a_n S + y S$  and so we may replace y by ysy and  $z_i$  by  $z_i sy$  without loss. Of course it is possible that some  $z_i = 0$  but in this case we can simply apply the inductive hypothesis to  $\{a_j, z_j, y: j \neq i\}$ . So assume that  $z_i \neq 0$  for  $1 \leq i \leq n-1$ . The net result of this is that, for  $1 \leq i \leq n-1$ , we now have  $r-ann(z_i) \supseteq r-ann(y)$  and hence  $udim(z_iS) \leq udim(yS) = 1$ . Hence  $z_iS$  is uniform for  $1 \leq i \leq n-1$ .

Set  $J = r-ann(a_n)$ . By Lemma 2.1(1) we can pick  $\mu \in S$  such that  $y\mu S \cap a_n S = 0$  and  $y\mu J \neq 0$ . By part (2) of that lemma, udim $(a_n + y\mu)S >$  udim $(a_n S)$ . Choose central elements  $\{\lambda_j \in Z(S): 1 \leq j \leq n\}$  that satisfy condition  $(*_n)$ . For any such  $\lambda_j$ , consider  $\alpha = a_n + y\mu\lambda_j$ . Since  $\lambda_j$  is a central regular element, the fact that  $y\mu J \neq 0$  implies that  $y\mu\lambda_j J \neq 0$ . Similarly,  $y\mu\lambda_jS \cap a_nS \subseteq y\mu S \cap a_nS = 0$ . Thus Lemma 2.1(2) can still be applied to ensure that

$$udim(a_n + y\mu\lambda_j)S > udim(a_nS) \quad \text{for each } \lambda_j.$$
(2.3)

Now, for some fixed  $1 \le i \le n - 1$ , consider the elements  $\gamma_{\ell} = a_i + z_i \mu \lambda_{\ell}$  for  $1 \le \ell \le n$ . We claim that  $\gamma_{\ell} \notin C_S(0)$  for at most one of these *n* elements. In order to prove this, it suffices to prove that, after relabelling, if  $\gamma_1 \notin C_S(0)$ , then  $\gamma_2 \in C_S(0)$ .

So, assume that  $K = r-ann_{S}(\gamma_{1}) \neq 0$  and write  $\gamma_{2} = \gamma_{1} + \delta$  for  $\delta = z_{i}\mu(\lambda_{2} - \lambda_{1})$ . We want to apply Lemma 2.1(2) to the elements  $f = \gamma_{1}$  and  $g = \delta$ . First, observe that if  $\delta K = 0$ , then  $z_{i}\mu K = 0$  since  $\lambda_{2} - \lambda_{1} \in C_{S}(0)$ , and so  $z_{i}\mu\lambda_{1}K = 0$ . Therefore,  $a_{i}K = 0$ , contradicting the fact that  $a_{i} \in C_{S}(0)$ . So  $\delta K \neq 0$ . In particular, as  $z_{i}S$  is uniform, so is  $\delta S$ .

Next suppose that  $\delta S \cap \gamma_1 S \neq 0$ . Then  $z_i \mu S \cap \gamma_1 S \neq 0$  and hence  $z_i \mu Q \cap \gamma_1 Q \neq 0$ , for Q = Q(S). But as  $z_i \mu S$  is uniform,  $z_i \mu Q$  is simple, whence  $z_i \mu \lambda_1 Q \subseteq z_i \mu Q \subseteq \gamma_1 Q$  and hence  $\gamma_1 Q = a_i Q + z_i \mu \lambda_1 Q = Q$ , by the regularity of  $a_i$ . This contradicts the fact that  $\gamma_1$  is not regular and implies that  $\delta S \cap \gamma_1 S = 0$ .

The hypotheses of Lemma 2.1(2) are therefore satisfied and, by that result,  $udim(\gamma_2 S) > udim(\gamma_1 S)$ . Moreover, as  $z_i \mu \lambda_1 S$  is uniform and  $a_i \in C_S(0)$ ,

$$\operatorname{udim}(\gamma_1 S) \ge \operatorname{udim}(a_i) - \operatorname{udim}(z_i \mu \lambda_1 S) \ge \operatorname{udim}(S) - 1.$$

Thus  $udim(\gamma_2 S) \ge udim(S)$  and  $\gamma_2$  is (right) regular, proving the claim.

Therefore, for each *i* there is at most one  $\lambda_{j(i)}$  with  $a_i + z_i \mu \lambda_{j(i)} \notin C_S(0)$ . Hence there is one  $\lambda = \lambda_j$  for  $1 \le j \le n$  such that  $a_i + z_i \mu \lambda \in C_S(0)$  for  $1 \le i \le n - 1$ . By (2.3), the lemma holds for this choice of  $\lambda$ .  $\Box$ 

**Proof of Theorem 1.4.** First, pick  $z \in I \cap C_S(0)$ . If we find e = ze' that satisfies (1.6), then automatically  $e \in I$ . In other words, replacing  $b_i$  by  $b_i z$  for all  $1 \leq i \leq n$ , it suffices to find  $e \in S$  that satisfies (1.6).

Either by Lemma 2.1 and induction, or by [7, Lemma 1.1], the theorem does hold for n = 1. By induction on n, we can find  $e \in S$  such that  $a_i + b_i e \in C_S(0)$  for  $1 \le i \le n - 1$ . Among such e choose the one for which  $udim(a_n + b_n e)S$  is as large as possible. If  $(a_n + b_n e) \in C_S(0)$  we are done, so assume not. Replace  $a_i$  by  $a_i + b_i e$  for all  $1 \le i \le n$ ; in particular  $a_i \in C_S(0)$  for  $1 \le i \le n - 1$ .

Now pick x by Lemma 2.1(1), for  $u = a_n$  and  $v = b_n$ . Set  $y = b_n x$  and  $z_i = b_i x$  for  $1 \le i \le n - 1$ ; thus yS is uniform with  $a_n S \cap yS = 0$  and so  $udim(a_n S + yS) > udim(a_n S)$ . Then Lemma 2.2 implies that we can find  $\lambda \in S$  such that  $a_i + z_i \lambda = a_i + b_i x \lambda \in C_S(0)$  for  $1 \le i \le n - 1$  while  $udim(a_n + b_n x \lambda)S = udim(a_n + y \lambda)S > udim(a_n S)$ . This contradicts the inductive hypothesis and proves the theorem.  $\Box$ 

Theorem 1.2 follows easily:

**Proof of Theorem 1.2.** This is similar to the proof of [6, Corollary 2.5]. By [6, Corollary 2.3] there exist prime ideals  $P_1, \ldots, P_n$  of R such that  $C_R(0) = \bigcap C_R(P_j)$ . We may assume that the  $P_j$  are distinct and we order them so that  $P_\ell \not\subseteq P_j$  for  $j > \ell$ . By induction suppose that we have found  $e \in R$  such that  $a_i + b_i e \in \bigcap_{j=1}^{r-1} C_R(P_j)$  for  $1 \le i \le n$ . (For r = 1 this assertion is vacuously true.) Replace  $a_i$  by  $a_i + b_i e$  for  $1 \le i \le n$  and set  $l' = \bigcap_{j=1}^{r-1} P_j$ , with l' = R if r = 1.

We now want to apply Theorem 1.4 to  $S = R/P_r$ , with  $I = (I' + P_r)/P_r$  and the images of  $a_i, b_i$ . As  $C_R(0) = \bigcap C_R(P_j) \subseteq C(P_r)$ , condition (1.5) does hold in *S*. Also  $I \neq 0$  by the ordering of the  $P_j$ . Finally, pick { $\lambda_i$ :  $1 \leq i \leq n$ } that satisfy ( $*_n$ ). By the choice of the  $P_j$ , again, the elements [ $\lambda_i + P_r$ ] still satisfy ( $*_n$ ) in *S*.

Thus the hypotheses of Theorem 1.4 hold and we can find  $e \in I'$  such that each  $a_i + b_i e \in C_R(P_r)$ . Since  $e \in I' \subseteq P_j$  for j < r, we see that  $a_i + b_i e \equiv a_i$  modulo  $P_j$  for these j. Hence  $a_i + b_i e \in \bigcap_{j=1}^r C_R(P_j)$  for  $1 \leq i \leq n$ . Thus, the theorem follows by induction and the choice of the  $P_j$ .  $\Box$ 

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