## Algebra

# Generating regular elements 

## Engendrer des éléments réguliers

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## A R T I C L E I N F O

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#### Abstract

Let $R$ be a prime right Goldie ring. A useful fact is that, if $a, b \in R$ are such that $a R+$ $b R$ contains a regular element, then there exists $\lambda \in R$ such that $a+b \lambda$ is regular. We show that the analogous result holds for $n \geqslant 1$ pairs of elements: if $R$ contains a field of cardinality at least $n+1$, and if $a_{i}, b_{i} \in R$ are such that $a_{i} R+b_{i} R$ contains a regular element for $1 \leqslant i \leqslant n$, then there exists a single element $\lambda \in R$ such that $a_{i}+b_{i} \lambda$ is regular for each $i$.


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## R É S U M É

Soit $R$ un anneau de Goldie premier. Un résultat utile est que si $a, b \in R$ sont tels que, $a R+b R$ contienne un élément régulier, alors il existe $\lambda \in R$ tel que $a+b \lambda$ est régulier. Nous montrons qu'un résultat analogue est vrai pour $n \geqslant 1$ paires de tels élément : si $R$ contient un corps de cardinal $>n$ et si les $a_{i}, b_{i} \in R$ sont tels que $a_{i} R+b_{i} R$ contienne un élément régulier, alors il existe $\lambda \in R$ tel que $a_{i}+b_{i} \lambda$ est régulier pour tout $i$.
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## 1. Introduction

Let $R$ be either a prime right Goldie ring or a noetherian ring. A useful fact is that, if $a, b \in R$ are such that $a R+b R$ contains a regular element, then some $a+b \lambda$ is regular (see [7, Lemma 1.1], respectively [6, Corollary 2.5]). Recently, Carpentier, De Sole and Kac raised the question of whether an analogous result holds for two or more pairs of elements. This was needed specifically for their paper [2] and is related to their work on differential operators and Poisson structures [1,3,4]. The aim of this note is to prove just such a result (see Theorems 1.2 and 1.4).

It is easy to see that this sort of result fails without some condition on $R$ (see Remark 1.3), and so we need the following hypothesis. We write $\mathcal{C}_{R}(I)$ or just $\mathcal{C}(I)$ for the set of elements of a ring $R$ that become regular modulo an ideal $I$.

Definition 1.1. Let $n \in \mathbb{N}$. Then $R$ satisfies $\left(*_{n}\right)$ if there exist regular central elements $\lambda_{1}, \ldots, \lambda_{n}$ such that $\lambda_{j}-\lambda_{i}$ is regular for all $1 \leqslant i<j \leqslant n$.

Clearly $\left(*_{n}\right)$ holds if $R$ contains a central subfield $k$ of cardinality $|k|>n$. Similarly, when $R$ is prime, $\left(*_{n}\right)$ holds provided the centre of $R$ has cardinality $|Z(R)|>n$.

[^0]Theorem 1.2. Fix an integer $n>0$ and let $R$ be a noetherian ring that satisfies $\left(*_{n}\right)$. Let $\left\{a_{i}, b_{i}: 1 \leqslant i \leqslant n\right\} \subseteq R$ be such that

$$
\left(a_{i} R+b_{i} R\right) \cap \mathcal{C}_{R}(0) \neq \emptyset \quad \text { for } 1 \leqslant i \leqslant n
$$

Then there exists $e \in R$ such that

$$
a_{i}+b_{i} e \in \mathcal{C}_{R}(0) \quad \text { for } 1 \leqslant i \leqslant n .
$$

Remark 1.3. Some condition like $\left(*_{n}\right)$ is necessary for the theorem to hold. For example, the theorem fails for $R=\mathbb{Z} / n \mathbb{Z}$ with $a_{i}=[-i+n \mathbb{Z}]$ and $b_{i}=1$ for $1 \leqslant i \leqslant n$. Indeed, the result fails if $R$ even has $\mathbb{Z} / n \mathbb{Z}$ as a ring-theoretic summand.

If one only wants $a_{1}+b_{1} e \in \mathcal{C}_{R}(0)$ to hold then the theorem is proved in [6, Corollary 2.5]. To prove Theorem 1.2 we will apply similar techniques. The main case is when the ring is prime, in which case we will get the following slightly stronger result.

Theorem 1.4. Fix an integer $n>0$ and let $S$ be a prime right Goldie ring that satisfies $\left(*_{n}\right)$. Let I be a non-zero ideal of $S$ (possibly $I=S$ ).

Let $\left\{a_{i}, b_{i}: 1 \leqslant i \leqslant n\right\} \subseteq R$ be such that

$$
\begin{equation*}
\left(a_{i} S+b_{i} S\right) \cap \mathcal{C}_{S}(0) \neq \emptyset \quad \text { for } 1 \leqslant i \leqslant n \tag{1.5}
\end{equation*}
$$

Then there exists $e \in I$ such that

$$
\begin{equation*}
a_{i}+b_{i} e \in \mathcal{C}_{S}(0) \quad \text { for } 1 \leqslant i \leqslant n \tag{1.6}
\end{equation*}
$$

## 2. The proofs

We begin by recalling various definitions and results, from [5, Chapter 2]. Let $S$ be a prime right Goldie ring with right Goldie quotient ring $Q=Q(S)$. A right $S$-module $M$ is uniform if $M \neq 0$ and every non-zero submodule $N \subseteq M$ is essential in $M$. The uniform dimension, udim $(M)$ is the maximum integer $n$ such that $M$ contains a direct sum of $n$ non-zero submodules (or $\operatorname{udim}(M)=\infty$ if no bound exists). If $M \subseteq Q$ then $\operatorname{udim}(M)$ is the length of the $Q$-module $M Q \cong M \otimes_{S} Q$. Finally, for $a \in S$, $\operatorname{udim}(a S)=\operatorname{udim}(S) \Longleftrightarrow a$ is right regular $\Longleftrightarrow a$ is regular. Given $m \in M$, write $\mathrm{r}-\mathrm{ann}(m)=\{r \in R: m r=0\}$ for the right annihilator of $m$.

The following result expands upon [7, Lemma 1.1].
Lemma 2.1. Let $S$ be a prime right Goldie ring.
(1) Let $u, v \in S$ with $\operatorname{udim}(u S+v S)>\operatorname{udim}(u S)$. Set $J=r-a n n(u)$. Then there exists $x \in S$ such that $0 \neq v x S$ is uniform, $v x S \cap$ $u S=0$ and $v x J \neq 0$;
(2) Suppose that $f, g \in S$ are such that $g S$ is uniform, with $f S \cap g S=0$ and $g \cdot r-a n n(f) \neq 0$. (This holds, in particular, if $f=u$ and $g=v x$ in the notation of part (1).) Then $\operatorname{udim}(f+g) S>\operatorname{udim}(f S)$.

Proof. (1) As $u$ cannot be right regular, $J \neq 0$. Next, $v S$ must contain a cyclic, uniform right ideal $L=v y S$ with $L \cap u S=\emptyset$. Since $S$ is prime, $v y S J \neq 0$ and so we can pick $z \in S$ such that $v y z J \neq 0$. Now (1) holds with $x=y z$.
(2) Use the final 6 lines from the proof of [7, Lemma 1.1].

The following more technical lemma forms the heart of the proof of Theorem 1.4.
Lemma 2.2. Fix an integer $n \geqslant 1$ and let $S$ be a prime right Goldie ring for which $\left(*_{n}\right)$ holds. Let $a_{1}, \ldots, a_{n}, z_{1}, \ldots, z_{n-1}, y \in S$ be such that
(a) $a_{i} \in \mathcal{C}_{S}(0)$ for $1 \leqslant i \leqslant n-1$, while
(b) $y S$ is uniform with $\operatorname{udim}\left(a_{n} S+y S\right)>\operatorname{udim}\left(a_{n} S\right)$.

Then there exists $\lambda \in S$ such that
(1) $a_{i}+z_{i} \lambda \in \mathcal{C}_{S}(0)$ for $1 \leqslant i \leqslant n-1$ while
(2) $\operatorname{udim}\left(a_{n}+y \lambda\right) S>\operatorname{udim}\left(a_{n} S\right)$.

Proof. We assume by induction that the result is true when $n$ is replaced by $n-1$ (with the case $n=1$ being Lemma 2.1). As $S$ is prime there exists $s \in S$ such that $y s y \neq 0$. Since $y s y S$ is then essential in $y S$, it follows that ( $a_{n} S+y s y S$ ) is essential in $a_{n} S+y S$ and so we may replace $y$ by $y s y$ and $z_{i}$ by $z_{i} s y$ without loss. Of course it is possible that some $z_{i}=0$
but in this case we can simply apply the inductive hypothesis to $\left\{a_{j}, z_{j}, y: j \neq i\right\}$. So assume that $z_{i} \neq 0$ for $1 \leqslant i \leqslant n-1$. The net result of this is that, for $1 \leqslant i \leqslant n-1$, we now have $r-a n n\left(z_{i}\right) \supseteq \mathrm{r}$-ann $(y)$ and hence $\operatorname{udim}\left(z_{i} S\right) \leqslant \operatorname{udim}(y S)=1$. Hence $z_{i} S$ is uniform for $1 \leqslant i \leqslant n-1$.

Set $J=\mathrm{r}-\mathrm{ann}\left(a_{n}\right)$. By Lemma 2.1(1) we can pick $\mu \in S$ such that $y \mu S \cap a_{n} S=0$ and $y \mu J \neq 0$. By part (2) of that lemma, $\operatorname{udim}\left(a_{n}+y \mu\right) S>\operatorname{udim}\left(a_{n} S\right)$. Choose central elements $\left\{\lambda_{j} \in Z(S): 1 \leqslant j \leqslant n\right\}$ that satisfy condition ( $*_{n}$ ). For any such $\lambda_{j}$, consider $\alpha=a_{n}+y \mu \lambda_{j}$. Since $\lambda_{j}$ is a central regular element, the fact that $y \mu J \neq 0$ implies that $y \mu \lambda_{j} J \neq 0$. Similarly, $y \mu \lambda_{j} S \cap a_{n} S \subseteq y \mu S \cap a_{n} S=0$. Thus Lemma 2.1(2) can still be applied to ensure that

$$
\begin{equation*}
\operatorname{udim}\left(a_{n}+y \mu \lambda_{j}\right) S>\operatorname{udim}\left(a_{n} S\right) \quad \text { for each } \lambda_{j} \tag{2.3}
\end{equation*}
$$

Now, for some fixed $1 \leqslant i \leqslant n-1$, consider the elements $\gamma_{\ell}=a_{i}+z_{i} \mu \lambda_{\ell}$ for $1 \leqslant \ell \leqslant n$. We claim that $\gamma_{\ell} \notin \mathcal{C}_{S}(0)$ for at most one of these $n$ elements. In order to prove this, it suffices to prove that, after relabelling, if $\gamma_{1} \notin \mathcal{C}_{S}(0)$, then $\gamma_{2} \in \mathcal{C}_{S}(0)$.

So, assume that $K=\mathrm{r}-\mathrm{ann}_{S}\left(\gamma_{1}\right) \neq 0$ and write $\gamma_{2}=\gamma_{1}+\delta$ for $\delta=z_{i} \mu\left(\lambda_{2}-\lambda_{1}\right)$. We want to apply Lemma 2.1(2) to the elements $f=\gamma_{1}$ and $g=\delta$. First, observe that if $\delta K=0$, then $z_{i} \mu K=0$ since $\lambda_{2}-\lambda_{1} \in \mathcal{C}_{S}(0)$, and so $z_{i} \mu \lambda_{1} K=0$. Therefore, $a_{i} K=0$, contradicting the fact that $a_{i} \in \mathcal{C}_{S}(0)$. So $\delta K \neq 0$. In particular, as $z_{i} S$ is uniform, so is $\delta S$.

Next suppose that $\delta S \cap \gamma_{1} S \neq 0$. Then $z_{i} \mu S \cap \gamma_{1} S \neq 0$ and hence $z_{i} \mu Q \cap \gamma_{1} Q \neq 0$, for $Q=Q(S)$. But as $z_{i} \mu S$ is uniform, $z_{i} \mu Q$ is simple, whence $z_{i} \mu \lambda_{1} Q \subseteq z_{i} \mu Q \subseteq \gamma_{1} Q$ and hence $\gamma_{1} Q=a_{i} Q+z_{i} \mu \lambda_{1} Q=Q$, by the regularity of $a_{i}$. This contradicts the fact that $\gamma_{1}$ is not regular and implies that $\delta S \cap \gamma_{1} S=0$.

The hypotheses of Lemma 2.1(2) are therefore satisfied and, by that result, $\operatorname{udim}\left(\gamma_{2} S\right)>\operatorname{udim}\left(\gamma_{1} S\right)$. Moreover, as $z_{i} \mu \lambda_{1} S$ is uniform and $a_{i} \in \mathcal{C}_{S}(0)$,

$$
\operatorname{udim}\left(\gamma_{1} S\right) \geqslant \operatorname{udim}\left(a_{i}\right)-\operatorname{udim}\left(z_{i} \mu \lambda_{1} S\right) \geqslant \operatorname{udim}(S)-1
$$

Thus $\operatorname{udim}\left(\gamma_{2} S\right) \geqslant \operatorname{udim}(S)$ and $\gamma_{2}$ is (right) regular, proving the claim.
Therefore, for each $i$ there is at most one $\lambda_{j(i)}$ with $a_{i}+z_{i} \mu \lambda_{j(i)} \notin \mathcal{C}_{S}(0)$. Hence there is one $\lambda=\lambda_{j}$ for $1 \leqslant j \leqslant n$ such that $a_{i}+z_{i} \mu \lambda \in \mathcal{C}_{S}(0)$ for $1 \leqslant i \leqslant n-1$. By (2.3), the lemma holds for this choice of $\lambda$.

Proof of Theorem 1.4. First, pick $z \in I \cap \mathcal{C}_{S}(0)$. If we find $e=z e^{\prime}$ that satisfies (1.6), then automatically $e \in I$. In other words, replacing $b_{i}$ by $b_{i} z$ for all $1 \leqslant i \leqslant n$, it suffices to find $e \in S$ that satisfies (1.6).

Either by Lemma 2.1 and induction, or by [7, Lemma 1.1], the theorem does hold for $n=1$. By induction on $n$, we can find $e \in S$ such that $a_{i}+b_{i} e \in \mathcal{C}_{S}(0)$ for $1 \leqslant i \leqslant n-1$. Among such $e$ choose the one for which udim $\left(a_{n}+b_{n} e\right) S$ is as large as possible. If $\left(a_{n}+b_{n} e\right) \in \mathcal{C}_{S}(0)$ we are done, so assume not. Replace $a_{i}$ by $a_{i}+b_{i} e$ for all $1 \leqslant i \leqslant n$; in particular $a_{i} \in \mathcal{C}_{S}(0)$ for $1 \leqslant i \leqslant n-1$.

Now pick $x$ by Lemma 2.1(1), for $u=a_{n}$ and $v=b_{n}$. Set $y=b_{n} x$ and $z_{i}=b_{i} x$ for $1 \leqslant i \leqslant n-1$; thus $y S$ is uniform with $a_{n} S \cap y S=0$ and so $\operatorname{udim}\left(a_{n} S+y S\right)>\operatorname{udim}\left(a_{n} S\right)$. Then Lemma 2.2 implies that we can find $\lambda \in S$ such that $a_{i}+z_{i} \lambda=$ $a_{i}+b_{i} \times \lambda \in \mathcal{C}_{S}(0)$ for $1 \leqslant i \leqslant n-1$ while $\operatorname{udim}\left(a_{n}+b_{n} x \lambda\right) S=\operatorname{udim}\left(a_{n}+y \lambda\right) S>\operatorname{udim}\left(a_{n} S\right)$. This contradicts the inductive hypothesis and proves the theorem.

Theorem 1.2 follows easily:
Proof of Theorem 1.2. This is similar to the proof of [6, Corollary 2.5]. By [6, Corollary 2.3] there exist prime ideals $P_{1}, \ldots, P_{n}$ of $R$ such that $\mathcal{C}_{R}(0)=\bigcap \mathcal{C}_{R}\left(P_{j}\right)$. We may assume that the $P_{j}$ are distinct and we order them so that $P_{\ell} \nsubseteq P_{j}$ for $j>\ell$. By induction suppose that we have found $e \in R$ such that $a_{i}+b_{i} e \in \bigcap_{j=1}^{r-1} \mathcal{C}_{R}\left(P_{j}\right)$ for $1 \leqslant i \leqslant n$. (For $r=1$ this assertion is vacuously true.) Replace $a_{i}$ by $a_{i}+b_{i} e$ for $1 \leqslant i \leqslant n$ and set $I^{\prime}=\bigcap_{j=1}^{r-1} P_{j}$, with $I^{\prime}=R$ if $r=1$.

We now want to apply Theorem 1.4 to $S=R / P_{r}$, with $I=\left(I^{\prime}+P_{r}\right) / P_{r}$ and the images of $a_{i}, b_{i}$. As $C_{R}(0)=\bigcap \mathcal{C}_{R}\left(P_{j}\right) \subseteq$ $\mathcal{C}\left(P_{r}\right)$, condition (1.5) does hold in $S$. Also $I \neq 0$ by the ordering of the $P_{j}$. Finally, pick $\left\{\lambda_{i}: 1 \leqslant i \leqslant n\right\}$ that satisfy $\left(*_{n}\right)$. By the choice of the $P_{j}$, again, the elements $\left[\lambda_{i}+P_{r}\right]$ still satisfy $\left(*_{n}\right)$ in $S$.

Thus the hypotheses of Theorem 1.4 hold and we can find $e \in I^{\prime}$ such that each $a_{i}+b_{i} e \in \mathcal{C}_{R}\left(P_{r}\right)$. Since $e \in I^{\prime} \subseteq P_{j}$ for $j<r$, we see that $a_{i}+b_{i} e \equiv a_{i}$ modulo $P_{j}$ for these $j$. Hence $a_{i}+b_{i} e \in \bigcap_{j=1}^{r} \mathcal{C}_{R}\left(P_{j}\right)$ for $1 \leqslant i \leqslant n$. Thus, the theorem follows by induction and the choice of the $P_{j}$.

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