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Harmonic Analysis/Dynamical Systems

Methods of harmonic analysis in nonlinear dynamics

Méthodes d'analyse harmonique en dynamique non linéaire

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ABSTRACT

For a pair of conjugate trigonometric polynomials $C(t) = \sum_{j=1}^{N} a_j \cos jt$, $S(t) = \sum_{j=1}^{N} a_j \sin jt$, normalized by the condition $\sum_{j=1}^{N} a_j = 1$, the following extremal value is found:

$$\sup_{a_1,\ldots,a_N} \min_t \{C(t)\colon S(t) = 0\} = -\tan^2 \frac{\pi}{2(N+1)}.$$

An application of this result in the control theory for nonlinear discrete systems is shown.

RÉSUMÉ

Pour un couple de polynômes trigonométriques $C(t) = \sum_{j=1}^{N} a_j \cos jt$, $S(t) = \sum_{j=1}^{N} a_j \sin jt$, normalisés par la condition $\sum_{j=1}^{N} a_j = 1$, on a la formule extrémale suivante :

$$\sup_{a_1,\ldots,a_N} \min_t \{C(t)\colon S(t) = 0\} = -\tan^2 \frac{\pi}{2(N+1)}.$$

On donne une application de ce résultat en théorie du contrôle à des systèmes non linéaires discrets.

1. Motivation

The problem of optimal influence on a chaotic regime is the most fundamental in nonlinear dynamics (cf. [2,7] or [5] for recent updates). Namely, let us consider an open scalar nonlinear discrete system:

$$x_{n+1} = f_h(x_n), \quad x_n \in \mathbb{R}^1, \ n = 1, 2, \ldots, \tag{1}$$

with a non-stable equilibrium point x^*. It is assumed that the function f_h depends on a finite number of parameters h and that for every admissible set of these parameters the function is defined and differentiable on a certain bounded interval and maps it into itself. The equilibrium point x^* and a multiplier $\mu = (f_h)'(x^*)$ are dependent on the parameters. It is assumed that $\mu \in (-\mu^*, -1]$, $\mu^* > 1$ and that the phenomenon of quasi-dynamical chaos is observed. The problem under

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investigation is to suppress the chaos by stabilizing the equilibrium x^* for all admissible values of the parameters using the control:

$$u = -\sum_{j=1}^{N-1} \varepsilon_j \big(f_h(x_{n-j+1}) - f_h(x_{n-j})\big), \quad 0 < \varepsilon_j < 1,\ j = 1, \dots, N-1, \tag{2}$$

in such a way that the depths of the used prehistory $N^* = N - 1$ will be minimal. Note that synchronization makes the control (2) to be zero, i.e. the closed system takes a control-free form. That means that the equilibria for the unclosed and closed systems are the same.

2. Statement of the problem

The characteristic polynomial for the linear part of the closed system (1) and (2) has the form:

$$\lambda^N + k\big(a_1\lambda^{N-1} + \dots + a_N\big), \tag{3}$$

where $a_1 = 1 - \varepsilon_1$, $a_j = \varepsilon_{j-1} - \varepsilon_j$, $j = 2, \dots, N-1$, $a_N = \varepsilon_{N-1}$, $k = |\mu|$. It is clear that $a_1 + \dots + a_n = 1$.

A change in parameter k could produce series of bifurcations in the system, leading to the appearance of a chaotic attractor. The first bifurcation value of the parameter is related to the momentum of losing stability. This value is related with the region of Schur's stability for the polynomials (3) in the space of parameter k. For $k = 0$, all the polynomials are stable. The zeros of the polynomials continuously depend on the parameter. Therefore, with some ($k_1 > 0$, $k_2 > 0$) if $k \in (-k_1, k_2)$, then the family (3) is stable, while for $k = k_2 + \varepsilon$ or $k = -k_1 - \varepsilon$ instability occurs.

It is a point of interest to find a direction determined by the vector of the coefficients $(a_1, \dots, a_n)$ for which the length of the connected component of the stability region will be maximal. In other words, we would like to maximize the length of robust stability $k_1 + k_2$ and to find the precise values of k_1 and k_2. Finding the minimal value for N^* is the dual problem to finding the maximal value of $k_1 + k_2$.

3. Connection with harmonic analysis

Given the interval $(-\mu^*, -1)$ both problems reduce to determining the minimal power of a polynomial and its coefficients in such a way that all zeros will be within the unit disc $\{|z| < 1\}$ for any $k \in (0, |\mu^*|)$.

Since on the boundary of the stability region:

$$\frac{1}{k} + \sum_{j=1}^{N} a_j e^{-\mathrm{i}jt} = 0 \tag{4}$$

we need to evaluate the quantity:

$$J_N = \sup_{a_1+\dots+a_n=1}\left[\min_t\left\{\Re\left(\sum_{j=1}^{N} a_j e^{-\mathrm{i}jt}\right):\ \Im\left(\sum_{j=1}^{N} a_j e^{-\mathrm{i}jt}\right) = 0\right\}\right]. \tag{5}$$

Note that $J_N \leqslant 0$. The relation (4) implies that the condition $|J_N| \cdot k < 1$ guaranties the stability of the polynomials (3) for all $k \in (0, |\mu^*|)$. With a little help from harmonic analysis, one can check that the choice $a_j = \frac{1}{N}$, $j = 1, \dots, N$ gives us the estimate $|J_N| \leqslant \frac{1}{N}$. So, we have got a fundamental fact – *given the interval $(-\mu^*, -1)$ one can stabilize the equilibrium in (1) using the prehistory of the length of order $|\mu^*|$*. Surprisingly, it turns out that the uniform choice of the coefficients is not the optimal one.

4. Theorem. *The following statements are valid*:

(i) $$J_N = -\tan^2\frac{\pi}{2(N+1)} \sim -\frac{1}{N^2}; \qquad N^* = \left[\frac{\pi}{2\cot^{-1}\sqrt{\mu^*}}\right] - 1.$$

(ii) *The interval $(-k_1, k_2)$ has the form $(-1, \frac{1}{|J_N|})$ and is of length $\csc^2\frac{\pi}{2(N+1)}$.*

(iii) *The optimal coefficients are unique and are defined by the formulas*:

$$\varepsilon_j = \sum_{k=j+1}^{N} a_k, \quad j = 1, \dots, N-1,$$

where

$$a_j = 2\cdot\tan\frac{\pi}{2(N+1)}\cdot\left(1 - \frac{j}{N+1}\right)\cdot\sin\frac{\pi j}{N+1}, \quad j = 1, \dots, N.$$

5. Example. Let $f_h : [0,1] \to [0,1]$ be a one-parameter logistic map:

$$f_h(x) = h \cdot x \cdot (1-x), \quad 0 \leqslant h \leqslant 4.$$

For $h \in (3,4]$, the equilibrium point $x^* = 1 - \frac{1}{h}$ is not stable and the multiplier $\mu \in [-2,-1)$. Therefore, $\frac{\pi}{2\cot^{-1}\sqrt{2}} \approx 2.55$ and the minimal depth of prehistory with the delayed feedback is $N^* = 1$. The optimal strength coefficient $\varepsilon_1^0 = \frac{1}{3}$, and the optimal control is $u = -\frac{1}{3}(f_h(x_n) - f_h(x_{n-1}))$, i.e. the closed system $x_{n+1} = f_h(x_n) + u$ has stable equilibrium points for $h \in (3,4]$.

6. The sketch of the proof

(The whole proof can be found in [3].) Let:

$$C(t) = \sum_{j=1}^{N} a_j \cos jt \quad \text{and} \quad S(t) = \sum_{j=1}^{N} a_j \sin jt.$$

We need to solve the following extremal problem. Find:

$$\rho \equiv \sup_{a_1+\cdots+a_N=1} \min_{-\pi \leqslant t \leqslant \pi} \{C(t) \colon S(t) = 0\}.$$

Since $S(t)$ is an odd function and $C(t)$ is an even function, then the above minimum can be taken over the set $0 \leqslant t \leqslant \pi$.

Let T denote the set of points where $S(t)$ changes sign, and let:

$$\rho_1 = \sup_{a_1+\cdots+a_N=1} \min_{0 \leqslant t \leqslant \pi} \{C(t) \colon t \in T \cup \{\pi\}\}.$$

It is not difficult to show that the supremum is achieved, i.e. there is a pair of optimal polynomials $C^0(t)$ and $S^0(t)$ such that:

$$\rho_1 = \min_{0 \leqslant t \leqslant \pi} \{C^0(t) \colon t \in T^0 \cup \{\pi\}\}.$$

It turns out, and it is *a difficult part of the proof*, that $T^0 = \emptyset$; therefore, $\rho_1 = C^0(\pi)$ and $S^0(t) \geqslant 0$ for $0 < t < \pi$.

Now, let us use the following presentation of $S(t)$ which is crucial:

$$S(t) = \sin t \cdot (\gamma_1 + 2\gamma_2 \cos t + \cdots + 2\gamma_N \cos(N-1)t).$$

Here $\gamma_s = \sum a_j$, and the summation runs on indices $s \leqslant j \leqslant N$ of same parity with s. This implies that $\gamma_1 + \gamma_2 = \sum_{j=1}^{N} a_j = 1$ and $C(\pi) = \gamma_2 - \gamma_1$. So,

$$\rho_1 = \max_{\gamma_1,\ldots,\gamma_N} \{-\gamma_1 + \gamma_2 \colon \gamma_1 + \gamma_2 = 1,\ S(t)/\sin t \geqslant 0,\ 0 < t < \pi\}.$$

Since $S(t)/\sin t$ is a non-negative even trigonometric polynomial, the well-known Fejér inequality [4] (see also [6, 6.7, Problem 52]) implies that:

$$|\gamma_2| \leqslant \cos\frac{\pi}{N+1} \cdot |\gamma_1| \tag{6}$$

therefore

$$\rho_1 \leqslant \rho_2 = \max_{\gamma_1,\gamma_2} \left\{-\gamma_1 + \gamma_2 \colon \gamma_1 + \gamma_2 = 1,\ |\gamma_2| \leqslant \cos\frac{\pi}{N+1} \cdot |\gamma_1|\right\}.$$

The maximum in ρ_2 is achieved for:

$$\gamma_1^0 = \frac{1}{1+\cos\frac{\pi}{N+1}}, \qquad \gamma_2^0 = \frac{\cos\frac{\pi}{N+1}}{1+\cos\frac{\pi}{N+1}}, \tag{7}$$

and is equal to:

$$\rho_2 = -\frac{1-\cos\frac{\pi}{N+1}}{1+\cos\frac{\pi}{N+1}} = -\tan^2\frac{\pi}{2(N+1)}.$$

Since γ_1^0 and γ_2^0 turn the Fejér inequality (6) into an equality, there is (a unique) trigonometric polynomial S^0 with the coefficients determined by γ_1^0 and γ_2^0 such that $S^0(t)/\sin t \geqslant 0$, $0 < t < \pi$, therefore $\rho_1 = \rho_2$. Now, let:

$$a_1^\varepsilon = \frac{a_1^0+\varepsilon}{1+\varepsilon}, \qquad a_j^\varepsilon = \frac{a_j^0}{1+\varepsilon}, \quad j \geqslant 2, \qquad S^\varepsilon(t)=\sum_{j=1}^N a_j^\varepsilon \sin(jt), \qquad C^\varepsilon(t)=\sum_{j=1}^N a_j^\varepsilon \cos(jt).$$

Since $a_1^\varepsilon+\cdots+a_N^\varepsilon=1$ one can conclude that $\rho \geqslant \min_{0\leqslant t\leqslant\pi}\{C^\varepsilon(t)\colon\ S^\varepsilon(t)=0\}$. It is easy to verify that $S^\varepsilon(t)=\frac{S^0(t)}{1+\varepsilon}+\frac{\varepsilon}{1+\varepsilon}\sin t$, $C^\varepsilon(t)=\frac{C^0(t)}{1+\varepsilon}+\frac{\varepsilon}{1+\varepsilon}\cos t$; therefore, for all $t\in(0,\pi)$ and $\varepsilon>0$, we have $S^\varepsilon(t)>0$. Since $S^\varepsilon(\pi)=0$, we have:

$$\rho \geqslant C^\varepsilon(\pi)=\frac{C^0(\pi)}{1+\varepsilon}-\frac{\varepsilon}{1+\varepsilon}$$

and letting $\varepsilon\to 0^+$, one gets $\rho \geqslant C^0(\pi)=-\tan^2\frac{\pi}{2(n+1)}=\rho_1$. On the other hand, $\min\{C(t)\colon\ t\in T\cup\{\pi\}\}\geqslant \min\{C(t)\colon\ S(t)=0\}$; therefore $\rho_1\geqslant\rho$, so $\rho=\rho_1$, while $\rho_1=\rho_2$, which proves (i). The statement (ii) is a direct consequence of (i).

Finally, starting from (7) one can get (iii). Similar computations can be found in [1].

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