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Algebraic Geometry

A smooth surface of tame representation type [☆]*Une surface lisse de type de représentation modéré*Daniele Faenzi ^a, Francesco Malaspina ^b^a Université de Pau et des pays de l'Adour, BP 576, 64012 Pau cedex, France^b Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

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ABSTRACT

We show that the Segre product of a line and a smooth conic, naturally embedded in \mathbb{P}^5 , is a smooth projective surface of tame representation type, namely all continuous families of indecomposable ACM bundles have dimension one. To our knowledge, this is the first example of smooth projective variety of this kind, besides the elliptic curve, which is of tame representation type according to Atiyah (1957).

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R É S U M É

Nous montrons que le produit d'une droite et d'une conique lisse, plongé dans \mathbb{P}^5 par Segre, est une variété projective de type modéré, autrement dit qu'il n'y a sur cette variété que des familles de dimension 1 au plus de fibrés indécomposables ACM. À notre connaissance, il s'agit du premier exemple de variété lisse projective de type modéré, mise à part la courbe elliptique, qui est de ce type d'après le travail fondamental d'Atiyah (1957).

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1. Introduction

Let Y be a smooth connected n -dimensional projective variety over an algebraically closed field k , with $n \geq 1$, embedded in \mathbb{P}^N by a very ample divisor class, denoted by h . Let R_Y be the homogeneous graded coordinate ring of Y , and assume that R_Y is a Cohen–Macaulay ring, so that Y is an ACM variety. A rank- r vector bundle E on Y is said to be *arithmetically Cohen–Macaulay (ACM)* if its module of global sections is a maximal Cohen–Macaulay module over R_Y . This is equivalent to the condition:

$$H_*^i(Y, E) := \bigoplus_{t \in \mathbb{Z}} H^i(Y, E(th)) = 0, \quad \text{for each } i = 1, \dots, n-1.$$

The variety Y is said to be of *finite representation type*, or *finite CM type* if it supports, up to twist by $\mathcal{O}_Y(th)$ and isomorphism, only a finite number of indecomposable ACM bundles. One must be aware that varieties of finite CM type are classified, see [6]. Their list is: linear embeddings of projective spaces and of smooth quadrics, rational normal curves, a smooth cubic scroll in \mathbb{P}^4 , and the Veronese surface in \mathbb{P}^5 .

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When Y supports continuous families of indecomposable ACM bundles of a given rank r , all non-isomorphic to one another, we say that Y is of *tame representation type* if there is a finite number of such families, and each of them has dimension at most one (and this for all r). On the other hand, Y is of *wild representation type* if there are ℓ -dimensional families of non-isomorphic indecomposable ACM sheaves, for arbitrarily large ℓ . In fact, only bundles of rank ≥ 2 matter here.

Here we announce that the surface $X = \mathbb{P}^1 \times \mathbb{P}^1$, embedded in \mathbb{P}^5 by the linear system $|\mathcal{O}_X(h)|$ of bidegree $(1, 2)$, is of tame representation type. Full proofs will be given in [7].

Theorem 1. *Up to a twist by $\mathcal{O}_X(th)$, any indecomposable non-zero ACM bundle on X is either a line bundle \mathcal{O}_X , $\mathcal{O}_X(-1, 0)$ or $\mathcal{O}_X(-1, -1)$, or an extension of the form:*

$$0 \rightarrow \mathcal{O}_X(0, -1)^{\oplus a} \rightarrow E \rightarrow \mathcal{O}_X(-1, 1)^{\oplus b} \rightarrow 0, \quad (1)$$

with either $|a - b| = 1$, in which case E is rigid, or $a = b \geq 1$, in which case the deformations of E are parameterized by a projective line.

In particular, fixed an even integer r , there is only one family of non-isomorphic indecomposable ACM bundles on X of rank r , parameterized by a projective line. If $r \geq 3$ is odd, there are only two non-isomorphic indecomposable ACM bundles on X of rank r , and these bundles are exceptional.

To our knowledge, this is the first example of smooth variety of tame representation type besides the elliptic curve, which in turn is of this kind according to the early work of Atiyah [1], see also [5]. The goal of [7] is to show that, except this example and the well-known cases of finite type, all other embeddings of homogeneous spaces are of wild representation type. This generalizes [4], where the same result is proved for linear embeddings of Segre varieties. Note that homogeneous spaces include all varieties of finite representation type, with the only exception of the cubic scroll in \mathbb{P}^4 . Anyway all other rational ACM surfaces in \mathbb{P}^4 are of wild representations type, see [10]. According to [9], the same happens to all rational normal scrolls besides the smooth quadric surface and our example (actually the argument of [9] does not apply to our surface, but fails only in this case). This leads to suspect that there are no other smooth projective positive-dimensional varieties of tame representation type, besides our example and the elliptic curve.

Also, in many interesting cases one seeks families of Ulrich bundles, namely ACM bundles E on Y such that the module of global sections of E has the highest possible number of generators, i.e. $\text{rank}(E) \deg(Y)$. It turns out that all the indecomposable ACM bundles on X with rank ≥ 2 are Ulrich bundles.

2. A family of ACM bundles parametrized by a projective line

Let X be the smooth projective surface obtained as the product of a line and a conic. So $X = \mathbb{P}^1 \times \mathbb{P}^1$ is embedded in \mathbb{P}^5 by the linear system $\mathcal{O}_X(h) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2)$, i.e. X is the rational normal scroll of type $(2, 2)$. We study ACM bundles E on X , i.e. bundles E on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $H^1(E(t, 2t)) = 0$, for any t .

As a preliminary step, let us list the ACM line bundles on X . Up to twist by $\mathcal{O}_X(t, 2t)$, they are

$$\mathcal{O}_X(0, -1), \quad \mathcal{O}_X(-1, 1), \quad \mathcal{O}_X(-1, 0), \quad \mathcal{O}_X(-1, -1), \quad \mathcal{O}_X(-1, -2).$$

The most important family of indecomposable ACM bundles on X is given by bundles fitting into (1), for integers $a, b \geq 0$ with $(a, b) \neq (0, 0)$. These are classified by the following proposition, whose proof relies on the Kronecker–Weierstrass classification of matrix pencils.

Proposition 2. *Let E be a bundle fitting in (1). Then E is an Ulrich bundle and, if E is indecomposable, then $\text{Ext}_X^2(E, E) = 0$ and $|a - b| \leq 1$. Further:*

- (i) if $a = b \pm 1$ then E is indecomposable if and only if E is exceptional, and in this case E corresponds to a general element of $\text{Ext}_X^1(\mathcal{O}_X(-1, 1)^{\oplus b}, \mathcal{O}_X(0, -1)^{\oplus a})$;
- (ii) if $a = b$ and E is indecomposable then E varies in a projective line. In fact, for all partitions $(\lambda_1, \dots, \lambda_r)$ of a , there are extensions of the above form such that E splits as the direct sum of r indecomposable bundles E_i , with $\text{rank}(E_i) = 2\lambda_i$, and each E_i varying in a \mathbb{P}^1 .

3. Generators of the derived category adapted to ACM bundles

We consider the collection of line bundles over X : $E_3 = \mathcal{O}_X$, $E_2 = \mathcal{O}_X(0, -1)$, $E_1 = \mathcal{O}_X(-1, -1)$, $E_0 = \mathcal{O}_X(-1, -2)$. It is easy to see that these line bundles form a full strongly exceptional collection on X , so that the derived category $\mathcal{D}^b(X)$ of bounded complexes of coherent sheaves on X is given by

$$\mathcal{D}^b(X) = \langle E_0, E_1, E_2, E_3 \rangle.$$

This choice of the generators of the derived category is adapted to the study of ACM bundles, as we shall see in a minute. We can compute the dual collection (F_0, F_1, F_2, F_3) of (E_0, E_1, E_2, E_3) . We get

$$F_3 = \mathcal{O}_X, \quad F_2 = \mathcal{O}_X(0, -1), \quad F_1 = \mathcal{O}_X(-1, 1)[-1], \quad F_0 = \mathcal{O}_X(-1, 0)[-1].$$

Given a vector bundle E over X , we construct a Beilinson complex quasi-isomorphic to E , by calculating $H^i(E \otimes E_j) \otimes F_j$, with $i, j \in \{0, 1, 2, 3\}$. Now, if E is ACM, then we have $H^1(E \otimes E_0) = 0$ and $H^1(E \otimes E_3) = 0$. Set $a = h^1(E \otimes E_2)$ and $b = h^1(E \otimes E_1)$. We get the following table:

*	*	*	*
0	b	a	0
*	*	*	*
E_0	E_1	E_2	E_3

The central line of this table can be thought of as a distinguished triangle:

$$F_1^{\oplus b} \rightarrow F_2^{\oplus a} \rightarrow E' \rightarrow F_1^{\oplus b}[1],$$

or in other words as an exact sequence $0 \rightarrow \mathcal{O}_X(0, -1)^{\oplus a} \rightarrow E' \rightarrow \mathcal{O}_X(-1, 1)^{\oplus b} \rightarrow 0$, which is the extension of the form (1). Looking more closely into the complex obtained by the above table, and using the fact that F_1 and F_3 are totally orthogonal to each other, as well as F_0 and F_2 , one can show that:

Lemma 3. *The bundle E' is a direct summand of E .*

By the lemma, we get another ACM bundle $E'' = E/E'$, that moreover satisfies the vanishing $H^1(E''(0, -1)) = H^1(E''(-1, -1)) = 0$. Any indecomposable direct summand of E' moves in a family of dimension ≤ 1 according to Proposition 2.

4. Splitting off line bundles from ACM bundles

In view of the above argument, to show that X is of tame representation type, it suffices to check that E'' splits as a direct sum of line bundles. To prove this fact we use the following notion, given in [3]:

Definition 4. A coherent sheaf F on $Q_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ is (p, p') -Qregular if

$$H^1(F(p-1, p')) = H^1(F(p, p'-1)) = H^2(F(p-1, p'-1)) = 0.$$

We often say *regular* instead of $(0, 0)$ -regular, p -regular instead of (p, p) -regular, and *irregular* for not regular. This notion of regularity coincides with the definition of p -Qregularity on X if $p = p'$ (see [2]) and agrees with the definition of (p, p') -regularity on $\mathbb{P}^1 \times \mathbb{P}^1$ by Hoffman and Wang (see [8]). Moreover, if F is regular, then it is globally generated (gg for short) and $F(p, p')$ is regular for $p, p' \geq 0$ (see [3, Proposition 2.2 and Remark 2.3]). A vector bundle becomes regular (respectively irregular) when twisted with $\mathcal{O}_X(t, 2t)$ for $t \gg 0$ (respectively $t \ll 0$).

Lemma 5. *Let F be an irregular indecomposable ACM bundle on X , with $F(1, 2)$ regular. If $H^1(F(-1, -1)) = H^1(F(0, -1)) = 0$, then F is isomorphic to $\mathcal{O}_X(-1, 0)$, or $\mathcal{O}_X(-1, -1)$ or $\mathcal{O}_X(-1, -2)$.*

To prove the lemma, we have to use the exact sequences:

$$0 \rightarrow F(s-1, t) \rightarrow F(s, t)^{\oplus 2} \rightarrow F(s+1, t) \rightarrow 0, \tag{2}$$

$$0 \rightarrow F(s, t-1) \rightarrow F(s, t)^{\oplus 2} \rightarrow F(s, t+1) \rightarrow 0. \tag{3}$$

Since F is irregular and $H^1(F(0, -1)) = 0$, we have $H^1(F(-1, 0)) \neq 0$ or $H^2(F(-1, -1)) \neq 0$. Let us look at these two cases.

1. If $H^2(F(-1, -1)) \neq 0$, we study two sub-cases, according to whether $F(1, 1)$ is regular or not.
 - 1.1. If $F(1, 1)$ is regular, then it is gg. Since $H^2(F(-1, -1)) \cong H^0(F^\vee(-1, -1)) \neq 0$, a general global section of $F(1, 1)$ thus splits off \mathcal{O}_X from $F(1, 1)$. But F is indecomposable so $F \cong \mathcal{O}_X(-1, -1)$.
 - 1.2. If the bundle $F(1, 1)$ is irregular, we have $H^1(F(0, 1)) \neq 0$, or $H^1(F(1, 0)) \neq 0$ or $H^2(F) \neq 0$. The last condition cannot occur, since $H^2(F) \cong H^0(F(-2, -2)) \neq 0$, so since $F(2, 2)$ is gg again $F \cong \mathcal{O}_X(-2, -2)$, which contra-

dicts that $F(1, 2)$ be regular. Moreover, F is ACM so $H^1(F) = 0$ and $F(1, 2)$ is regular so $H^1(F(0, 2)) = 0$, hence $H^1(F(0, 1)) = 0$ from (3) with $s = 0, t = 1$.

So we must have $H^1(F(1, 0)) \neq 0$. Now, $F(1, 2)$ is regular so $H^1(F(1, 1)) = 0$ hence from (3) with $s = t = 1$ we get $H^0(F(1, 2)) \neq 0$. On the other hand, $H^2(F) = H^1(F) = 0$ so from (2) with $s = t = 0$ we get $H^2(F(-1, 0)) \neq 0$. Hence $H^0(F^\vee(-1, -2)) \neq 0$, and again $F \cong \mathcal{O}_X(-1, -2)$.

2. It remains to look at the case $H^1(F(-1, 0)) \neq 0$, and we can now assume $H^2(F(-1, -1)) = 0$. Again from (2) with $s = t = 0$ we obtain $H^0(F(1, 0)) \neq 0$. Also, using $H^1(F(-1, -1)) = H^2(F(-1, -1)) = 0$, from (3) with $s = t = -1$ we get $H^2(F(-1, -2)) \neq 0$. Then $H^0(F^\vee(-1, 0)) \neq 0$, so $F \cong \mathcal{O}_X(-1, 0)$.

Let us check that this gives [Theorem 1](#). Given an indecomposable ACM bundle E on X , iterating [Lemma 3](#) for all twists $E(t, 2t)$, we see that either $E(t, 2t)$ fits into (1) for some t or E satisfies $H^1(E(t, 2t - 1)) = H^1(E(t - 1, 2t - 1)) = 0$ for all t . In the former case, the statement is contained in [Proposition 2](#). In the latter case, we choose the smallest integer t such that $E(t + 1, 2t + 2)$ is regular. Applying [Lemma 5](#) to $E(t, 2t)$, we get that E is $\mathcal{O}_X(-1, 0)$, or $\mathcal{O}_X(-1, -1)$ or \mathcal{O}_X up to twist. [Theorem 1](#) follows.

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