C. R. Acad. Sci. Paris, Ser. I 351 (2013) 381–383

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Algebraic Geometry

# Stable bundles as Frobenius morphism direct image

## *Faisceaux stables en tant qu'images directes par le morphisme de Frobenius*

Congjun Liu, Mingshuo Zhou

*Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, PR China*

ARTICLE INFO

*Article history:*
Received 27 January 2013
Accepted after revision 26 April 2013
Available online 4 June 2013

Presented by Claire Voisin

ABSTRACT

Let $X$ be a smooth projective curve of genus $g \geqslant 2$ over an algebraically closed field $k$ of characteristic $p > 0$, and let $F: X \to X_1$ be the relative Frobenius morphism. We show that a vector bundle $E$ on $X_1$ is the direct image under $F$ of some stable bundle on $X$ if and only if the instability of $F^*E$ is equal to $(p-1)(2g-2)$.



RÉSUMÉ

Soient $X$ une courbe projective lisse de genre $g \geqslant 2$ définie sur un corps $k$ algébriquement clos de caractéristique $p > 0$, et $F: X \to X_1$ le morphisme de Frobenius relatif. On montre qu'un fibré vectoriel $E$ sur $X_1$ est l'image directe sous $F$ d'un certain fibré stable sur $X$ si et seulement si l'instabilité de $F^*E$ est égale à $(p-1)(2g-2)$.



## 1. Introduction

Let $X$ be a smooth projective curve of genus $g \geqslant 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$. The absolute Frobenius morphism $F_X: X \to X$ is induced by $\mathcal{O}_X \to \mathcal{O}_X$, $f \mapsto f^p$. Let $F: X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over $k$. One of the themes is to study its action on the geometric objects on $X$. Recall that a vector bundle $E$ on a smooth projective curve is called semi-stable (resp. stable) if $\mu(E') \leqslant \mu(E)$ (resp. $\mu(E') < \mu(E)$) for any nontrivial proper subbundle $E' \subset E$, where $\mu(E)$ is the slope of $E$. It is known that $F_*$ preserves the stability of vector bundles (cf. [5]), but $F^*$ does not preserve the semi-stability of vector bundles (cf. [1] for example).

Semi-stable bundles are basic constituents of vector bundles in the sense that any bundle $E$ admits a unique filtration:

$$\mathrm{HN}_\bullet(E): \quad 0 = \mathrm{HN}_0(E) \subset \mathrm{HN}_1(E) \subset \cdots \subset \mathrm{HN}_\ell(E) = E,$$

which is the so-called Harder–Narasimhan filtration, such that:

(1) $\mathrm{gr}_i^{\mathrm{HN}}(E) := \mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)$ $(1 \leqslant i \leqslant \ell)$ are semi-stable;
(2) $\mu(\mathrm{gr}_1^{\mathrm{HN}}(E)) > \mu(\mathrm{gr}_2^{\mathrm{HN}}(E)) > \cdots > \mu(\mathrm{gr}_\ell^{\mathrm{HN}}(E))$.

The rational number $\mathrm{I}(E) := \mu(\mathrm{gr}_1^{\mathrm{HN}}(E)) - \mu(\mathrm{gr}_\ell^{\mathrm{HN}}(E))$, which measures how far a vector bundle is from being semi-stable, is called the instability of $E$. It is clear that $E$ is semi-stable if and only if $\mathrm{I}(E) = 0$.

*E-mail addresses:* liucongjun@amss.ac.cn (C. Liu), zhoumingshuo@amss.ac.cn (M. Zhou).

1631-073X/$ – see front matter 
http://dx.doi.org/10.1016/j.crma.2013.04.021

Given a semi-stable bundle $E$ on $X_1$, then $F^*E$ may not be semi-stable, so it is natural to consider the instability $\mathrm{I}(F^*E)$. In [4, Theorem 3.1], the author proves $\mathrm{I}(F^*E) \leqslant (\ell - 1)(2g - 2)$, where $\ell$ is the length of Harder–Narasimhan filtration of $F^*E$. If $E = F_*W$ where $W$ is a stable bundle on $X$, we know, by Sun's theorem [5, Theorem 2.2], that $E$ is stable, the length of Harder–Narasimhan filtration of $F^*E$ is $p$ and $\mathrm{I}(F^*E) = (p - 1)(2g - 2)$. Thus $\mathrm{I}(F^*E) = (p - 1)(2g - 2)$ is a necessary condition for $E$ to be a direct image under Frobenius. In this short note, we show the following theorem:

**Theorem 1.** *Let $E$ be a stable vector bundle on $X$. Then the following statements are equivalent*:

(1) *There exists a stable bundle $W$ such that $E = F_*W$*;
(2) $\mathrm{I}(F^*E) = (p - 1)(2g - 2)$.

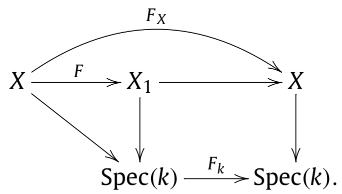
The case $\mathrm{rk}\, E = p$ was proved in [3]. Our observation is that the arguments in [3] together with Sun's theorem imply the general case.

## 2. Proof of the theorem

Let $X$ be a smooth projective curve over an algebraically closed field $k$ with $\mathrm{char}(k) = p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by the following homomorphism:

$$\mathcal{O}_X \to \mathcal{O}_X, \quad f \mapsto f^p.$$

Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over $k$ that satisfies the following commutative diagram:




For a vector bundle $E$ on $X$, the slope of $E$ is defined as

$$\mu(E) := \frac{\deg E}{\mathrm{rk}\, E}$$

where $\mathrm{rk}\, E$ (resp. $\deg E$) denotes the rank (resp. degree) of $E$. Then:

**Definition 1.** A vector bundle $E$ on $X$ is called semi-stable (resp. stable) if for any nontrivial proper subbundle $E' \subset E$, we have

$$\mu(E') \leqslant (\text{resp.} <)\, \mu(E).$$

**Theorem 2** *(Harder–Narasimhan filtration). For any vector bundle $E$, there is a unique filtration*:

$$\mathrm{HN}_\bullet(E): \quad 0 = \mathrm{HN}_0(E) \subset \mathrm{HN}_1(E) \subset \cdots \subset \mathrm{HN}_\ell(E) = E,$$

*which is called Harder–Narasimhan filtration, such that*:

(1) $\mathrm{gr}_i^{\mathrm{HN}}(E) := \mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)$ $(1 \leqslant i \leqslant \ell)$ *are semi-stable*;
(2) $\mu(\mathrm{gr}_1^{\mathrm{HN}}(E)) > \mu(\mathrm{gr}_2^{\mathrm{HN}}(E)) > \cdots > \mu(\mathrm{gr}_\ell^{\mathrm{HN}}(E))$.

By using this unique filtration of $E$, an invariant $\mathrm{I}(E)$ of $E$, which is called the instability of $E$ was introduced (see [5] and [4]). It is a rational number and measures how far is $E$ from being semi-stable.

**Definition 2.** Let $\mu_{\max}(E) = \mu(\mathrm{gr}_1^{\mathrm{HN}}(E))$, $\mu_{\min}(E) = \mu(\mathrm{gr}_\ell^{\mathrm{HN}}(E))$. Then the instability of $E$ is defined to be

$$\mathrm{I}(E) := \mu_{\max}(E) - \mu_{\min}(E).$$

It is easy to see that a vector bundle $E$ is semi-stable if and only if $\mathrm{I}(E) = 0$.

For any semi-stable bundle $E$, let

$$\mathrm{HN}_\bullet(F^*E): \quad 0 = \mathrm{HN}_0(F^*E) \subset \mathrm{HN}_1(F^*E) \subset \cdots \subset \mathrm{HN}_\ell(F^*E) = F^*E$$

be the Harder–Narasimhan filtration of $F^*E$. Then we have the following lemma, which is implicit in [3].

**Lemma 1.** *For any semi-stable bundle $E$, we have*

$$\mu_{\max}(F^*E) \leqslant p \cdot \mu(E) + (p-1)(g-1);$$
$$\mu_{\min}(F^*E) \geqslant p \cdot \mu(E) - (p-1)(g-1),$$

*and if* $\mathrm{I}(F^*E) = \mu_{\max}(F^*E) - \mu_{\min}(F^*E) = (p-1)(2g-2)$. *Then*

$$\mu_{\max}(F^*E) = p \cdot \mu(E) + (p-1)(g-1);$$
$$\mu_{\min}(F^*E) = p \cdot \mu(E) - (p-1)(g-1).$$

Now we prove our theorem by using this Lemma 1, the canonical filtration on the vector bundle $V = F^*F_*W$ and Sun's theorem on the stability of Frobenius' direct images.

**Proof of Theorem 1.** (1) ⇒ (2). In [2, Section 5.3], there is a canonical filtration on the vector bundle $V = F^*F_*W$:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell \subset \cdots \subset V_{p-1} \subset V_p = V,$$

which is indeed the Harder–Narasimhan filtration on $V$, and satisfies

$$V_\ell/V_{\ell-1} \cong (V_{\ell+1}/V_\ell) \otimes \Omega_X^1$$

for $1 \leqslant \ell \leqslant p-1$, and $V_p/V_{p-1} \cong W$. So $\mu(V_p/V_{p-1}) = \mu(W)$, $\mu(V_0/V_1) = \mu(W) + (p-1)(2g-2)$, and now the result is clear.

(2) ⇒ (1). Since $\mathrm{I}(F^*E) = (p-1)(2g-2)$, we have $\mu_{\max}(F^*E) = p \cdot \mu(E) + (p-1)(g-1)$, $\mu_{\min}(F^*E) = p \cdot \mu(E) - (p-1)(g-1)$ by Lemma 1. We consider the surjection:

$$F^*E \to \mathrm{gr}_\ell^{\mathrm{HN}}(F^*E).$$

The bundle $\mathrm{gr}_\ell^{\mathrm{HN}}(F^*E)$ is semi-stable of slope $\mu_{\min}(F^*E)$. Replacing $\mathrm{gr}_\ell^{\mathrm{HN}}(F^*E)$ by a stable graded piece $W$ in the Jordan–Hölder filtration of $\mathrm{gr}_\ell^{\mathrm{HN}}(F^*E)$, we have a surjection:

$$F^*E \to W,$$

where $W$ is a stable bundle of slope $\mu(W) = \mu_{\min}(F^*E) = p \cdot \mu(E) - (p-1)(g-1)$. By adjunction, we have a nontrivial morphism:

$$\psi : E \to F_*W.$$

By Sun's theorem (cf. [5, Theorem 2.2]), we know that $F_*W$ is a stable bundle of slope:

$$\mu(F_*W) = \frac{\mu(W)}{p} + \frac{(p-1)(g-1)}{p} = \mu(E).$$

Thus $\psi$ induce an isomorphism:

$$E \cong F_*W. \qquad \square$$


## Acknowledgement

The authors would like to thank their advisor Professor Xiaotao Sun for encouragements and many useful discussions.


## References


[1] D. Gieseker, Stable vector bundles and the Frobenius morphism, Ann. Sci. Éc. Norm. Super. (4) 6 (1973) 95–101.
[2] K. Joshi, S. Ramanan, E. Xia, J.-K. Yu, On vector bundles destabilized by Frobenius pull-back, Compos. Math. 142 (3) (2006) 616–630.
[3] V. Mehta, C. Pauly, Semistability of Frobenius direct images over curves, Bull. Soc. Math. Fr. 135 (2007) 105–117.
[4] X. Sun, Remarks on semistability of G-bundles in positive characteristic, Compos. Math. 119 (1999) 41–52.
[5] X. Sun, Direct images of bundles under Frobenius morphism, Invent. Math. 173 (2008) 427–447.