Differential Geometry

Curvature properties of anti-Kähler–Codazzi manifolds

Propriétés de courbure des variétés anti-Kähler–Codazzi

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A B S T R A C T

In this paper we shall consider a new class of integrable almost anti-Hermitian manifolds, which will be called anti-Kähler–Codazzi manifolds, and we will investigate their curvature properties.

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R É S U M É

Dans cet article, nous allons considérer une nouvelle classe de variétés intégrables presque anti-hermitiennes qui seront appelées variétés anti-Kähler–Codazzi, et nous allons étudier les propriétés de courbure de ces variétés.

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1. Introduction

Let \((M, J)\) be a 2n-dimensional almost complex manifold, where \(J\) denotes its almost complex structure. A semi-Riemannian metric \(g\) of neutral signature \((n, n)\) is an anti-Hermitian (Norden) metric if:

\[ g(JX, Y) = g(X, JY) \]

for any \(X, Y \in \mathcal{H}(M)\), where \(\mathcal{H}(M)\) is the module of vector fields on \(M\). An almost complex manifold \((M, J)\) with an anti-Hermitian metric is referred to as an almost anti-Hermitian manifold. Structures of this kind have been also studied under the name: almost complex structures with pure (or B-) metric. An anti-Kähler (Kähler–Norden) manifold can be defined as a triple \((M, g, J)\), which consists of a smooth manifold \(M\) endowed with an almost complex structure \(J\) and an anti-Hermitian metric \(g\) such that \(\nabla J = 0\), where \(\nabla\) is the Levi-Civita connection of \(g\). It is well known that the condition \(\nabla J = 0\) is equivalent to \(C\)-holomorphy (analyticity) of the anti-Hermitian metric \(g\) [1], i.e. \(\Phi_J g = 0\), where \(\Phi_J\) is the Tachibana operator [4]:

\[ (\Phi_J g)(X, Y, Z) = (L_J g - L_X g)(Y, Z) \]

where \(G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z)\) is the twin anti-Hermitian metric. It is a remarkable fact that \((M, g, J)\) is anti-Kähler if and only if the twin anti-Hermitian structure \((M, G, J)\) is anti-Kähler. This is of special significance for anti-Kähler metrics since in such case \(g\) and \(G\) share the same Levi-Civita connection. Since in dimension 2 an anti-Kähler manifold is flat, we assume in the sequel that \(\dim M \geq 4\).

Let now \((M, g, J)\) be an almost anti-Hermitian manifold and let \(R(X, Y) = [\nabla X, \nabla Y] - \nabla_{[X,Y]}\) be the curvature operator of the Levi-Civita connection \(\nabla\) on \(M\). Then the Ricci tensor \(S\) is defined as \(S(X, Y) = \text{trace}(Z \mapsto R(Z, X)Y)\). We note that for the case where \((M, g, J)\) is anti-Kähler manifold these tensors have the following properties [1]:

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\[ J(R(X, Y)Z) = R(JX, Y)Z = R(X, JY)Z = R(X, Y)JZ, \quad S(JX, Y) = S(X, JY), \]
i.e. \( R \) and \( S \) are pure tensors with respect to the structure \( J \) (for more details about pure tensors, see [3]). Moreover, in such a manifold, \( R \) and \( S \) are \( C \)-holomorphic tensors.

2. Anti-Kähler–Codazzi manifolds

It is well known that the pair \((J, g)\) of an almost Hermitian structure defines a fundamental 2-form \( \Omega \) by \( \Omega(X, Y) = g(JX, Y) \). If the skew-symmetric tensor \( \Omega \) is a Killing–Yano tensor, i.e.

\[
(\nabla_X \Omega)(Y, Z) + (\nabla_Y \Omega)(X, Z) = 0
\]

or equivalently if the almost complex structure \( J \) satisfies \( (\nabla_X J)Y + (\nabla_Y J)X = 0 \) for any \( X, Y \in \mathfrak{X}(M) \), then the manifold is called a nearly Kähler manifold (or \( K \)-space).

Let now \((M, g, J)\) be an almost anti-Hermitian manifold. Then the pair \((J, g)\) defines, as usual, the twin anti-Hermitian metric \( G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z) \), but \( G \) is symmetric, rather than a 2-form. Thus, the anti-Hermitian pair \((J, g)\) does not give rise to a 2-form, and the Killing–Yano equation (1) has no immediate meaning. Therefore, we can replace the Killing–Yano equation by Codazzi equation:

\[
(\nabla_X G)(Y, Z) - (\nabla_Y G)(X, Z) = 0. \tag{2}
\]

Eq. (2) is equivalent to:

\[
(\nabla_X J)Y - (\nabla_Y J)X = 0. \tag{3}
\]

If the almost complex structure of almost anti-Hermitian manifold satisfies (3), then the triple \((M, J, g)\) is called an anti-Kähler–Codazzi manifold (or AKC-space).

Remark 1. Let the tensor \( G \) (i.e. the twin anti-Hermitian metric) be a Killing symmetric tensor, i.e.

\[
\sigma \, (\nabla_X G)(Y, Z) = 0,
\]

where \( \sigma \) is the cyclic sum with respect to \( X, Y \) and \( Z \). This is the class of the quasi-Kähler manifold with anti-Hermitian (Norden) metric [2].

Theorem 2.1. Anti-Kähler–Codazzi manifolds have integrable almost anti-Hermitian structures.

Proof. Using \( \nabla_X Y - \nabla_Y X = [X, Y] \), \( (\nabla_X J)(Y) = -J(\nabla_X J)Y \) for every almost anti-Hermitian manifold and (3), we have:

\[
N_J(X, Y) = [JX, YJ] - [JX, JY] - [JY, X] - [X, Y]
= \nabla_JX JY - \nabla_JY JX - \nabla(JX Y - JY X) - J(\nabla_JX Y - \nabla_JY X) + J^2(\nabla_X Y - \nabla_Y X)
= -J((\nabla_JX Y - (\nabla_JY J)X) + (\nabla_JY J)Y - (\nabla_JY J)X)
= -J((\nabla_JY J)X - (\nabla_JX J)Y) + J((\nabla_JY J)X - (\nabla_JX J)Y) = 0.
\]
i.e. the Nijenhuis tensor \( N_J \) vanishes. Conversely, from property \( N_J = 0 \) not conclude (3). The proof of the theorem is complete. □

3. Curvature properties

Let the triple \((M, g, J)\) be an anti-Kähler–Codazzi manifold. Since \( \nabla_X \) commutes with every contraction (trace) of a tensor field and trace \( J = 0 \), we have from (3):

\[
q = \text{trace} \{ V \rightarrow (\nabla_V J)X - (\nabla_X J)V \}
= \text{trace} \{ V \rightarrow (\nabla_V J)X \} - \nabla_X \text{trace} J
= \text{trace} \{ V \rightarrow (\nabla_V J)X \} = 0.
\]

Let \( x^1, \ldots, x^{2n} \) be a local coordinate system in \( M \). By setting \( V = \frac{\partial}{\partial x^i} \) and \( X = \frac{\partial}{\partial x^j} \), \( i, j = 1, \ldots, 2n \), in this equation, we have

\[
q_j = \nabla_i J^i_j = 0.
\]

Applying the Ricci identity to the tensor field \( J \), we find:

\[
\nabla_k \nabla_j J^h_l - \nabla_j \nabla_k J^h_l = R_{kij}^h J^i_l - R_{kji}^h J^i_l,
\]

where \( R_{kij}^h \) are components of curvature tensor \( R \). After contraction with respect to \( k \) and \( h \) in this equation, by virtue of \( q_j = 0 \), we have:
\[ \nabla_h \nabla_j J^h_i = S_{jt} J^t_i - R_{hji} J^h_i = S_{jt} J^t_i - R_{hji} g^{il} J^h_l = S_{jt} J^t_i - R_{hji} g^{il} J^l_i = S_{jt} J^t_i - R_{hji} J^h_i, \]

where \( S_{jt} \) are the components of the Ricci tensor \( S \), \( G^{lh} \) are the contravariant components of twin anti-Hermitian metric \( G \) and \( H_{ji} = R_{hji} g^{lh} \). Since \( G^{lh} = C^{hl} \), \( R_{(hji)} = 0 \), \( R_{(hji)} = 0 \), from \( H_{ji} = R_{hji} G^{lh} \) we have:

\[ H_{ji} = \frac{1}{2} (R_{hji} + R_{jhi} ) G^{lh} = \frac{1}{2} (R_{hji} + R_{jhi} ) G^{lh} \]

or

\[ H_{ji} = \frac{1}{2} (R_{hji} - R_{jhi} + R_{ihi} - R_{hij} ) G^{lh} = 0, \]

i.e. \( H \) is a symmetric tensor field. Then, by virtue of \( H_{jji} = 0 \), from (3) and (4), we have:

\[ S_{jt} J^t_i - S_{it} J^t_j = \nabla_h (\nabla_j J^h_i - \nabla_i J^h_j) = 0. \]

Since \( S_{ij} = S_{ji} \), from the last equation we have:

**Theorem 3.1.** In an anti-Kähler–Codazzi manifold, the Ricci tensor is pure with respect to the complex structure \( J \).

We now put:

\[ \hat{S}_{ji} = -H_{ji} J^h_i = -R_{hji} g^{lh} J^h_l. \]

We call \( \hat{S} \) the Ricci* tensor of \( M \). On the other hand, by virtue of \( \hat{S}_{ji} J^h_i = H_{ji} \) Eq. (4) can be written as:

\[ \nabla_h \nabla_j J^h_i = S_{jt} J^t_i - \hat{S}_{jt} J^t_i = \left( S_{jt} - \hat{S}_{jt} \right) J^t_i. \]

Hence, we have

**Theorem 3.2.** Let \((M, g, J)\) be an anti-Kähler–Codazzi manifold. In order to have \( S = \hat{S} \), it is necessary and sufficient that:

\[ \nabla_h \nabla_j J^h_i = 0, \]

where \( S \) and \( \hat{S} \) are the Ricci and Ricci* tensors, respectively.

From this theorem, we have:

**Corollary 3.3.** If an anti-Kähler–Codazzi manifold is anti-Kähler \((\nabla_j J^h_i = 0)\), then \( S = \hat{S} \).

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