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Differential Geometry

An entropy formula relating Hamilton's surface entropy and Perelman's $\mathcal{W}$ entropy

Une formule d'entropie reliant l'entropie de Hamilton des surfaces et l'entropie $\mathcal{W}$ de Perelman

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ABSTRACT

In this note, based on Hamilton's surface entropy formula, we construct an entropy formula of Perelman's type for the Ricci flow on a closed surface with positive curvature. Similar to Perelman's $\mathcal{W}$ entropy, the critical point of our entropy is the gradient shrinking soliton; however, there is no conjugate heat equation involved. This shows a close relation between Hamilton's entropy and Perelman's $\mathcal{W}$ entropy on closed surfaces.

RÉSUMÉ

Dans cette note, à partir de la formule de Hamilton pour l'entropie des surfaces, nous construisons une formule d'entropie de type Perelman pour le flot de Ricci sur une surface fermée à courbure positive. De même que pour l'entropie $\mathcal{W}$ de Perelman, le point critique de notre entropie est le soliton gradient décroissant, bien qu'il n'y ait pas ici d'équation de la chaleur qui soit mise en jeu. Ceci démontre une relation étroite entre l'entropie de Hamilton et l'entropie $\mathcal{W}$ de Perelman sur les surfaces fermées.

1. Introduction

Assume that M is a compact surface endowed with a 1-parameter family of Riemannian metrics $g(t)$. By the Gauss–Bonnet formula, we know that $\int_M R\,\mathrm{d}\mu \equiv 4\pi\chi(M)$, where $\mathrm{d}\mu$ is the area form with respect to $g(t)$. Thus if $(M, g(t))$ has positive curvature, which means that M is diffeomorphic to S^2 or $\mathbb{R}P^2$, then $R\,\mathrm{d}\mu/(4\pi\chi(M))$ can be regarded as a probability measure, and Hamilton's surface entropy [5] is defined by

$$\mathcal{N}\big(g(t)\big) = \int_M R\log R\,\mathrm{d}\mu. \tag{1.1}$$

Hamilton [5] showed that along the normalized Ricci flow, $\mathcal{N}(g(t))$ is non-increasing, and later Chow gave an alternative proof in [1]. One can also see [2,3] for this classical result.

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We focus on the un-normalized Ricci flow

$$\frac{\partial}{\partial t} g = -R\, g, \quad t \in (\alpha, T). \tag{1.2}$$

When $\alpha = -\infty$ the solution is called ancient solution.

Besides Hamilton's surface entropy (1.1), there is Perelman's $\mathcal{W}$ entropy [7] defined by

$$\mathcal{W}(g, u, \tau) \doteqdot \int_M \big(\tau\big(|\nabla \log u|^2 + R\big) - \log\big((4\pi\tau)^{n/2} u\big) - n\big) u \, d\mu \tag{1.3}$$

restricted to u satisfying $\int_M u \, d\mu = 1$ and $\tau > 0$. When g evolves along the Ricci flow (1.2), u satisfies the conjugate heat equation:

$$\frac{\partial u}{\partial t} = -\Delta u + Ru \tag{1.4}$$

and $\tau = T - t$, Perelman shows that:

$$\frac{d}{dt} \mathcal{W}(g, u, \tau) = \int_M 2\tau \left| \mathrm{Rc} - \nabla\nabla \log u - \frac{1}{2\tau} g \right|^2 u \, d\mu. \tag{1.5}$$

We are curious about the relation between Hamilton's surface entropy (1.1) and Perelman's $\mathcal{W}$ entropy (1.3). The first comment was made by Perelman. In [7, 5.3], he writes: "An entropy formula for the Ricci flow in dimension two was found by Chow; there seems to be no relation between his formula and ours."

Then in [6, Section 5], Ni asked the following question: "Is there any connection between Perelman's entropy formula and Hamilton's entropy formula at all?" And in [6, Addenda, Section 2], Ni constructed a dual entropy relating Perelman's entropy and Hamilton's surface entropy. Ni's entropy is called "dual" because its critical point is an expander, while the critical point of Perelman's entropy is a shrinker.

In this short note, based on Hamilton's surface entropy, we define a new entropy formula whose critical point is exactly the shrinking gradient soliton. This entropy formula shows a close relation between Perelman's entropy and Hamilton's surface entropy.

We now introduce the following definition.

Definition 1.1. Suppose $(M, g(t))$ is a compact surface with positive curvature. We define an entropy formula of $(M, g(t))$ by

$$\mathcal{E}\big(g(t)\big) \doteqdot \int_M \big(\tau\big(R - |\nabla \log R|^2\big) - \log R - \log\tau\big) R \, d\mu \tag{1.6}$$

where $\tau = T - t$.

On the sphere S^2 with canonical Ricci flow $g(t) = 2\tau g_{\mathrm{can}}$ where g_{can} is the standard round metric of radius 1, we have $R = 1/\tau$ and, moreover, $\mathcal{E}(g(t)) = 8\pi$.

We have:

Theorem 1.2. *Assume $(M, g(t))$ is a compact 2-dimensional solution to the Ricci flow (1.2) with positive curvature, then the entropy $\mathcal{E}$ satisfies*:

$$\frac{d}{dt} \mathcal{E}\big(g(t)\big) = \int_M 2\tau \left| \nabla\nabla \log R + \frac{R}{2} g - \frac{1}{2\tau} g \right|^2 R \, d\mu. \tag{1.7}$$

2. Calculations

This section is devoted to prove (1.7). We first prove that Hamilton's surface entropy $\mathcal{N}(g(t))$ is convex in time t along the un-normalized Ricci flow (1.2). This result should have been known to experts. Since we have not been able to find it explicitly in literature we present the calculations.

Lemma 2.1. *Assume $(M, g(t))$ is a compact 2-dimensional solution to the Ricci flow (1.2) with positive curvature, then the first two derivatives of the surface entropy $\mathcal{N}(g(t))$ are given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}\big(g(t)\big)=\int_M(\Delta\log R+R)R\,\mathrm{d}\mu=\int_M\big(R-|\nabla\log R|^2\big)R\,\mathrm{d}\mu \tag{2.8}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{N}\big(g(t)\big)=\int_M 2\left|\nabla\nabla\log R+\frac{R}{2}g\right|^2 R\,\mathrm{d}\mu. \tag{2.9}$$

Proof. The proof is a straightforward calculation. Notice that along the Ricci flow, one has

$$\frac{\partial}{\partial t}R=\Delta R+R^2,\qquad \frac{\partial}{\partial t}\mathrm{d}\mu=-R\,\mathrm{d}\mu,\qquad \frac{\partial}{\partial t}\Delta=R\Delta.$$

We calculate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}\big(g(t)\big)&=\int_M\big(\Delta R+R^2\big)\log R+\big(\Delta R+R^2\big)-R^2\log R\,\mathrm{d}\mu\\&=\int_M\Delta R\log R+R^2\,\mathrm{d}\mu\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{N}\big(g(t)\big)&=\int_M\left(\frac{\partial\Delta}{\partial t}R\right)\log R+\Delta\left(\frac{\partial R}{\partial t}\right)\log R+\Delta R\left(\frac{1}{R}\frac{\partial R}{\partial t}\right)+2R\frac{\partial R}{\partial t}-R\big(\Delta R\log R+R^2\big)\,\mathrm{d}\mu\\&=\int_M\Delta\big(\Delta R+R^2\big)\log R+\frac{(\Delta R)^2}{R}+3R\Delta R+R^3\,\mathrm{d}\mu.\end{aligned}$$

Now using integration by parts, we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{N}\big(g(t)\big)&=\int_M\big(\Delta R+R^2\big)\left(\frac{\Delta R}{R}-\frac{|\nabla R|^2}{R^2}\right)+\frac{(\Delta R)^2}{R}+3R\Delta R+R^3\,\mathrm{d}\mu\\&=\int_M 2\Delta R\cdot\frac{\Delta R}{R}-\Delta R\cdot|\nabla\log R|^2+5R\Delta R+R^3\,\mathrm{d}\mu\\&=\int_M 2\Delta R\cdot\Delta\log R+\Delta R\cdot|\nabla\log R|^2+5R\Delta R+R^3\,\mathrm{d}\mu\\&=\int_M -2\langle\nabla R,\nabla\Delta\log R\rangle+R\Delta|\nabla\log R|^2+5R\Delta R+R^3\,\mathrm{d}\mu.\end{aligned}$$

By the Bochner formula

$$\Delta|\nabla\log R|^2=2|\nabla\nabla\log R|^2+R|\nabla\log R|^2+2\langle\nabla\Delta\log R,\nabla\log R\rangle$$

we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{N}\big(g(t)\big)&=\int_M 2R|\nabla\nabla\log R|^2+R^2|\nabla\log R|^2+5R\Delta R+R^3\,\mathrm{d}\mu\\&=\int_M 2R|\nabla\nabla\log R|^2+2R\Delta R-2|\nabla R|^2+R^3\,\mathrm{d}\mu\\&=\int_M 2R\left(|\nabla\nabla\log R|^2+R\Delta\log R+\frac{R^2}{2}\right)\mathrm{d}\mu\\&=\int_M 2\left|\nabla\nabla\log R+\frac{R}{2}g\right|^2 R\,\mathrm{d}\mu.\qquad\square\end{aligned}$$

Now we are ready to prove our main theorem.

Proof of Theorem 1.2. By the same trick as in [4], we rewrite Eq. (2.9) purposely to fit the shrinking soliton equation.

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{N}\big(g(t)\big) &= \int_M 2R\left|\nabla\nabla\log R + \frac{R}{2}g\right|^2 d\mu \\ &= \int_M 2R\left|\nabla\nabla\log R + \frac{R}{2}g - \frac{1}{2\tau}g\right|^2 + \frac{2}{\tau}R(\Delta\log R + R) - \frac{R}{\tau^2}\,d\mu \\ &= \int_M 2R\left|\nabla\nabla\log R + \frac{R}{2}g - \frac{1}{2\tau}g\right|^2 d\mu + \frac{2}{\tau}\mathcal{N}'\big(g(t)\big) - \frac{4\pi\chi(M)}{\tau^2}.\end{aligned}$$

Hence we have

$$\begin{aligned}\int_M 2R\left|\nabla\nabla\log R + \frac{R}{2}g - \frac{1}{2\tau}g\right|^2 d\mu &= \mathcal{N}''\big(g(t)\big) - \frac{2}{\tau}\mathcal{N}'\big(g(t)\big) + \frac{4\pi\chi(M)}{\tau^2} \\ &= \frac{1}{\tau}\frac{d}{dt}\big(\tau\mathcal{N}' - \mathcal{N} - 4\pi\chi(M)\log\tau\big).\end{aligned}$$

The above calculations suggest to define $\mathcal{E}(g(t))$ to be $\tau\mathcal{N}' - \mathcal{N} - 4\pi\chi(M)\log\tau$ namely by Eq. (1.6), and then we have

$$\frac{d}{dt}\mathcal{E}\big(g(t)\big) = \int_M 2\tau\left|\nabla\nabla\log R + \frac{R}{2}g - \frac{1}{2\tau}g\right|^2 R\,d\mu,$$

and this proves Theorem 1.2. □

Remark 2.2. We note Perelman's formula (1.5) holds as long as u is a positive solution to the conjugate heat equation (1.4). If we do calculations for all positive solution to

$$\frac{\partial}{\partial t}u = \Delta u + Ru \tag{2.10}$$

we get

$$\left(\frac{\partial}{\partial t} - \Delta - R\right)(u\Delta\log u + uR) = 2u|\nabla\nabla\log u + \mathrm{Rc}|^2 + u^2\Delta\left(\frac{R}{u}\right). \tag{2.11}$$

Thus, only when $u = cR$ we can get an entropy formula of Perelman's type.

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