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Functional Analysis

Subnormality of 2-variable weighted shifts with diagonal core ☆

Sous-normalité de shifts pondérés à deux variables avec cœur diagonal

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ABSTRACT

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. Given a 2-variable weighted shift $\mathbf{T}$ with diagonal core, we prove that LPCS is soluble for $\mathbf{T}$ if and only if LPCS is soluble for some power $\mathbf{T}^{\mathbf{m}}$ ($\mathbf{m} \in \mathbb{Z}_+^2$, $\mathbf{m} \equiv (m_1, m_2)$, $m_1, m_2 \geqslant 1$). We do this by first developing the basic properties of diagonal cores, and then analyzing how a diagonal core interacts with the rest of the 2-variable weighted shift.

RÉSUMÉ

Le problème du relèvement des opérateurs sous-normaux commutatifs (LPCS) consiste à rechercher des conditions nécessaires ou suffisantes pour que deux opérateurs sous-normaux sur l'espace de Hilbert admettent des extensions normales commutatives. Étant donné un opérateur de décalage pondéré $\mathbf{T}$ à deux variables avec cœur diagonal, nous prouvons que le LPCS est résoluble pour $\mathbf{T}$ si et seulement si le LPCS est résoluble pour une certaine puissance $\mathbf{T}^{\mathbf{m}}$ ($\mathbf{m} \in \mathbb{Z}_+^2, \mathbf{m} \equiv (m_1, m_2)$, $m_1, m_2 \geqslant 1$). Nous le faisons en développant d'abord les propriétés de base des cœurs diagonaux, puis en analysant la façon dont un cœur diagonal interagit avec le reste de l'opérateur.

1. Introduction

For an operator T on Hilbert space, it is well known that the subnormality of T implies the subnormality of T^m ($m \geqslant 2$). The converse implication, however, is false; in fact, the subnormality of all powers T^m ($m \geqslant 2$) does not necessarily imply the subnormality of T, even if $T \equiv W_\omega$ is a unilateral weighted shift [15,16,20,21]. To study relevant generalizations of these results in the multivariable case, the standard starting assumptions on a pair $\mathbf{T} \equiv (T_1, T_2)$ are that $T_1 T_2 = T_2 T_1$ and that

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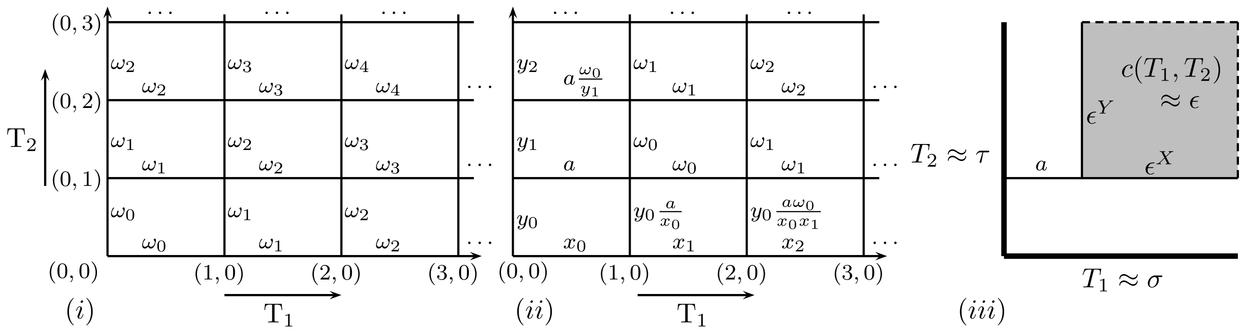

Fig. 1. Weight diagrams of $\Theta(W_\omega)$ and $W_{(\alpha,\beta)}$ with $c(W_{(\alpha,\beta)}) \cong \Theta(W_\omega)$; and Berger measure diagram of $W_{(\alpha,\beta)}$ with $c(W_{(\alpha,\beta)}) \cong \Theta(W_\omega)$, respectively.

Fig. 1. Schéma des poids de $\Theta(W_\omega)$ et $W_{(\alpha,\beta)}$ avec $c(W_{(\alpha,\beta)}) \cong \Theta(W_\omega)$; et schéma de la mesure de Berger de $W_{(\alpha,\beta)}$ avec $c(W_{(\alpha,\beta)}) \cong \Theta(W_\omega)$, respectivement.

each component T_i is subnormal ($i = 1, 2$). With this in hand, multivariable versions of the 1-variable results are highly nontrivial.

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. Single and multivariable weighted shifts have played an important role in the study of LPCS. They have also played a significant role in the study of cyclicity and reflexivity, in the study of C^*-algebras generated by multiplication operators on Bergman spaces, as fertile ground to test new hypotheses, and as canonical models for theories of dilation and positivity (cf. [5–7,9,11–13,17–20]). In our previous work we have studied LPCS from a number of different approaches. One such approach is to consider a commuting pair $\mathbf{T}$ of subnormal operators and to ask to what extent the existence of liftings for the powers $\mathbf{T}^{\mathbf{m}} := (T_1^{m_1}, T_2^{m_2})$ ($m_1, m_2 \geqslant 1$) can guarantee a lifting for $\mathbf{T}$. E. Franks proved in [14] that a commuting pair $\mathbf{T}$ is subnormal if and only if $p(T_1, T_2)$ is subnormal for all polynomials p in two variables of total degree at most 5. This result motivates the question of whether the subnormality of a pair of powers $(T_1^{m_1}, T_2^{m_2})$ can be used to establish the subnormality of $\mathbf{T}$ [10]. Given two bounded double-indexed sequences α and β we define the 2-*variable weighted shift* $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ acting on $\ell^2(\mathbb{Z}_+^2)$ by $T_1 e_{(k_1,k_2)} := \alpha_{(k_1,k_2)} e_{(k_1+1,k_2)}$ and $T_2 e_{(k_1,k_2)} := \beta_{(k_1,k_2)} e_{(k_1,k_2+1)}$ with $T_1T_2 = T_2T_1$, where $\{e_{(k_1,k_2)}\}$ denotes the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+^2)$. For the class of 2-variable weighted shifts, it is often the case that the powers are less complex than the initial pair; thus it becomes especially significant to unravel LPCS under the action $(m_1, m_2) \mapsto W_{(\alpha,\beta)}^{(m_1,m_2)} \equiv \mathbf{T}^{(m_1,m_2)}$ ($m_1, m_2 \geqslant 1$).

For the class of 2-variable weighted shifts with core of tensor form, denoted $\mathcal{TC}$, we showed in [12] that if $W_{(\alpha,\beta)} \in \mathcal{TC}$, then $W_{(\alpha,\beta)}$ is subnormal if and only if $W_{(\alpha,\beta)}^{(m_1,m_2)}$ is subnormal for some $m_1, m_2 \geqslant 1$. We thus characterized LPCS in terms of the above action in the class $\mathcal{TC}$. (The *core* $c(W_{(\alpha,\beta)})$ of a 2-variable weighted shift is the restriction of $W_{(\alpha,\beta)}$ to the subspace generated by $\{e_{(k_1,k_2)}\}_{k_1,k_2 \geqslant 1}$; we say that $c(W_{(\alpha,\beta)})$ is of tensor form if it is unitarily equivalent to $(I \otimes W_\epsilon, W_\nu \otimes I)$ for some unilateral weighted shifts W_ϵ and W_ν.)

In this paper, we study a new class, $\mathcal{DC}$, of multivariable weighted shifts, those with *diagonal* core. Put simply, a core of tensor form corresponds to a Berger measure of the form $\xi \times \eta$, while a diagonal core is associated to a Berger measure supported in the diagonal $\{(s, s) \in \mathbb{R}^2\colon\ s \geqslant 0\}$ (see Fig. 1(i)). The classes $\mathcal{TC}$ and $\mathcal{DC}$ share some properties, but not others. For example, restrictions of shifts in $\mathcal{DC}$ do remain in $\mathcal{DC}$, just as it happens for the class $\mathcal{TC}$. On the other hand, the power of a weighted shift in the class $\mathcal{TC}$ splits as a direct sum of shifts in $\mathcal{TC}$, while the same is not true for shifts with diagonal core. Thus, while LPCS is soluble in $\mathcal{TC}$ for $\mathbf{T}$ if and only if it is soluble for any power $\mathbf{T}^{\mathbf{m}}$, as we mentioned above, it is not a priori obvious whether the same result holds in the class $\mathcal{DC}$. Our main result establishes that this is indeed the case (see Section 3 below).

Given a 1-variable unilateral weighted shift W_ω associated with a weight sequence $\{\omega_k\}_{k=0}^\infty$, we embed ω into $\ell^2(\mathbb{Z}_+^2)$ as follows:

$$\alpha_{(k_1,k_2)} \equiv \beta_{(k_1,k_2)} := \omega_{k_1+k_2} \quad (k_1, k_2 \geqslant 0). \tag{1}$$

We denote the associated 2-variable weighted by $\Theta(W_\omega)$ (see Fig. 1(i)); we will soon see that $\Theta(W_\omega) \in \mathcal{DC}$. The *Berger measure* of W_ω, denoted by $\mu \equiv \mu[\omega]$, is the unique probability Borel measure compactly supported in $\mathbb{R}$ satisfying $\int s^n \, d\mu(s) = \gamma_n := \omega_0^2 \cdots \omega_{n-1}^2$ for all $n \geqslant 1$) [1, III.8.16]; γ_n is called the n-th moment of ω.

In Section 2, we first prove that the map Θ preserves many structural properties, like k-hyponormality and subnormality, and that the Berger measure of a subnormal W_ω transfers in a canonical way to $\Theta(W_\omega)$. Observe that 2-variable weighted shifts with diagonal core can be regarded as antipodal to those whose core is of tensor form, since the Berger measure for their (diagonal) core is supported in a "thin" set (the diagonal $\{(s, s)\colon\ s \in \mathbb{R}\}$), while for the other class the Berger measure is as "thick" as possible, that is, a Cartesian product.

We end this section by introducing some notation which will be needed later. We denote the class of commuting pairs of operators on Hilbert space by $\mathfrak{C}_0$, the class of subnormal pairs by $\mathfrak{C}_\infty \equiv \mathfrak{H}_\infty$, and for an integer $k \geqslant 1$, the class of k-hyponormal pairs in $\mathfrak{C}_0$ by $\mathfrak{C}_k$. We show that $\mathfrak{C}_\infty \subsetneq \cdots \subsetneq \mathfrak{C}_k \subsetneq \cdots \subsetneq \mathfrak{C}_1 \subsetneq \mathfrak{C}_0$ (Corollary 2.3). We also denote the class of commuting pairs of *subnormal* operators on Hilbert space by $\mathfrak{H}_0$, and for an integer $k \geqslant 1$, the class of k-hyponormal pairs in $\mathfrak{H}_0$ by $\mathfrak{H}_k$. For each integer $k \geqslant 0$, it is possible to prove that $\mathfrak{H}_k \subsetneq \mathfrak{C}_k$.

Let $\mathcal{R}_{(i,j)}(W_{(\alpha,\beta)})$ denote the restriction of $W_{(\alpha,\beta)}$ to $\mathcal{M}_i \cap \mathcal{N}_j$, where $\mathcal{M}_i := \bigvee\{e_{(k_1,k_2)}\colon k_1 \geqslant 0, k_2 \geqslant i\}$ and $\mathcal{N}_j := \bigvee\{e_{(k_1,k_2)}\colon k_1 \geqslant j, k_2 \geqslant 0\}$. The core of $W_{(\alpha,\beta)}$, $c(W_{(\alpha,\beta)})$ is $\mathcal{R}_{(1,1)}(W_{(\alpha,\beta)})$. We let $\mathcal{TC} := \{W_{(\alpha,\beta)} \in \mathfrak{H}_0\colon c(W_{(\alpha,\beta)})$ is of tensor form$\}$ and $\mathcal{DC} := \{W_{(\alpha,\beta)} \in \mathfrak{H}_0\colon c(W_{(\alpha,\beta)}) = \Theta(\mathrm{shift}(\alpha_{1,1}, \alpha_{2,1}, \alpha_{3,1}, \ldots))\}$. For a weighted sequence $\omega \equiv \{\omega_n\}_{n=0}^{\infty}$, the diagonal embedding (1) gives rise to a commuting 2-variable weighted shift $\Theta(W_\omega)$, as can be easily proved. For each $\mathbf{k} \in \mathbb{Z}_+^2$, we let $B_{\mathbf{k}} := \{W_{(\alpha,\beta)} \in \mathfrak{H}_0\colon \mathcal{R}_{\mathbf{k}}(W_{(\alpha,\beta)}) \in \operatorname{ran}\Theta\}$. Observe that for $\mathbf{k}, \mathbf{m} \in \mathbb{Z}_+^2$ with $\mathbf{k} \leqslant \mathbf{m}$ (i.e., $\mathbf{m} - \mathbf{k} \in \mathbb{Z}_+^2$), we have $B_{\mathbf{k}} \subseteq B_{\mathbf{m}}$. Thus, the collection $\{B_{\mathbf{k}}\}$ forms an ascending chain with respect to set inclusion and the partial order induced by $\mathbb{Z}_+^2$. Moreover, $\mathcal{DC} = B_{11} \subseteq B_{\mathbf{k}}$ for all $\mathbf{k} \in \mathbb{Z}_+^2$. We prove that for all $\mathbf{k} \in \mathbb{Z}_+^2$, $B_{\mathbf{k}} = \mathcal{DC}$. We do this by applying [5, Proposition 1.5], which states that in a subnormal unilateral weighted shift W_ω each weight ω_k ($k \geqslant 1$) is completely determined by the Berger measure of the restriction of W_ω to the invariant subspace generated by $\{e_{k+1}, e_{k+2}, \ldots\}$.

Given integers p and m ($m \geqslant 1, 0 \leqslant p \leqslant m-1$), consider $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+) = \bigvee\{e_n\colon n \geqslant 0\}$ and define $\mathcal{H}_p := \bigvee\{e_{m\ell+p}\colon \ell \geqslant 0\}$, so $\mathcal{H} = \bigoplus_{p=0}^{m-1} \mathcal{H}_p$. For a sequence $\omega \equiv \{\omega_n\}_{n=0}^{\infty}$, let $\omega(m:p) := \{\prod_{k=0}^{m-1} \omega_{m\ell+p+k}\}_{\ell=0}^{\infty}$. Then for $m \geqslant 1$ and $0 \leqslant p \leqslant m-1$, W_ω^m is unitarily equivalent to $\bigoplus_{p=0}^{m-1} W_{\omega(m:p)}$. It is well known that for a subnormal unilateral weighted shift W_ω with moments $\{\gamma_n\}_{n\geqslant 0}$ and Berger measure $\mathrm{d}\mu(s)$, $W_{\omega(m,p)}$ is subnormal with Berger measure $\mathrm{d}\mu_{(m,0)}(s) = \mathrm{d}\mu(s^{\frac{1}{m}})$ and $\mathrm{d}\mu_{(m,p)}(s) = \frac{s^{\frac{p}{m}}}{\gamma_p}\,\mathrm{d}\mu(s^{\frac{1}{m}})$ for $1 \leqslant p \leqslant m-1$ [4, Theorem 2.9]. By analogy with the above-mentioned one-variable decomposition, we split the ambient space $\ell^2(\mathbb{Z}_+^2)$ as an orthogonal direct sum $\bigoplus_{p=0}^{m_1-1}\bigoplus_{q=0}^{m_2-1} \mathcal{H}_{(p,q)}^{(m_1,m_2)}$, where $\mathcal{H}_{(p,q)}^{(m_1,m_2)} := \bigvee\{e_{(m_1k+p, m_2\ell+q)}\colon k, \ell \geqslant 0;\ 0 \leqslant p \leqslant m_1-1;\ 0 \leqslant q \leqslant m_2-1\}$. Each $\mathcal{H}_{(p,q)}^{(m_1,m_2)}$ reduces $T_1^{m_1}$ and $T_2^{m_2}$, and $\mathbf{T}^{\mathbf{m}}$ is subnormal if and only if each $\mathbf{T}^{\mathbf{m}}|_{\mathcal{H}_{(p,q)}}$ is subnormal, where $\mathbf{m} := (m_1, m_2)$. Our main result states that for 2-variable weighted shifts $\mathbf{T}$ with diagonal core, the subnormality of $\mathbf{T}^{\mathbf{m}}$ for some $\mathbf{m} \geqslant \mathbf{1} := (1,1)$ implies the subnormality of $\mathbf{T}$.

2. The canonical embedding of a unilateral weighted shift

In this section, we describe some basic results of the canonical embedding $\omega \longmapsto \Theta(\omega)$ defined by (1). We begin by listing several well-known results which will be needed in the sequel.

Lemma 2.1. (i) (*See* [2].) *Let $W_\omega e_i = \omega_i e_{i+1}$ ($i \geqslant 0$) be hyponormal. Then for $k \geqslant 1$, W_ω is k-hyponormal if and only if the Hankel matrix $H(k;n) := (\gamma_{n+i+j-2})_{i,j=1}^{k+1} \geqslant 0$ for all $n \geqslant 0$.*

(ii) (*See* [8].) *Given finite sets of positive real numbers $(x_i)_{i=1}^n$ and $(\omega_i)_{i=0}^k$, let $W_\omega := \mathrm{shift}(x_n, \ldots, x_1, (\omega_0, \ldots, \omega_k)^\wedge)$, where $\mathrm{shift}((\omega_0, \ldots, \omega_k)^\wedge)$ denotes the recursively generated weighted shift with initial weights $(\omega_0, \ldots, \omega_k)$ (see* [3, Section 3]*). Then W_ω is subnormal if and only if $\{W_\omega$ is $([\frac{k+1}{2}]+1)$-hyponormal (when $n=1$) or W_ω is $([\frac{k+1}{2}]+2)$-hyponormal (when $n>1$)$\}$.*

(iii) (*See* [9].) $W_{(\alpha,\beta)} \in \mathfrak{H}_k \iff M_{\mathbf{u}}(k) := (\gamma_{\mathbf{u}+(n,m)+(p,q)})_{\substack{0\leqslant n+m\leqslant k\\ 0\leqslant p+q\leqslant k}} \geqslant 0$ *(all $\mathbf{u} \in \mathbb{Z}_+^2$). (For $u_1, u_2 \geqslant 1$, $\gamma_{(u_1,u_2)} := \alpha_{(0,0)}^2 \cdots \alpha_{(u_1-1,0)}^2 \beta_{(u_1,0)}^2 \cdots \beta_{(u_1,u_2-1)}^2$, $\gamma_{(u_1,0)} := \alpha_{(0,0)}^2 \cdots \alpha_{(u_1-1,0)}^2$, $\gamma_{(0,u_2)} := \beta_{(0,0)}^2 \cdots \beta_{(0,u_2-1)}^2$ and $\gamma_{(0,0)} := 1$.)*

(iv) (*See* [6].) *Let μ be the Berger measure of a subnormal $W_{(\alpha,\beta)}$, and for $j \geqslant 0$ let ξ_j be the Berger measure of the associated j-th horizontal 1-variable weighted shift $W_{\alpha^{(j)}}$ for $W_{(\alpha,\beta)}$; i.e., $W_{\alpha^{(j)}} e_{(k_1,j)} := \alpha_{(k_1,j)} e_{(k_1+1,j)}$ ($k_1 \geqslant 0$). Then $\xi_j = \mu_j^X$ (the marginal measure of μ_j), where $\mathrm{d}\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j\,\mathrm{d}\mu(s,t)$; more precisely, $\mathrm{d}\xi_j(s) = \{\frac{1}{\gamma_{0j}} \int_Y t^j\,\mathrm{d}\Phi_s(t)\}\,\mathrm{d}\mu^X(s)$, where $\mathrm{d}\mu(s,t) \equiv \mathrm{d}\Phi_s(t)\,\mathrm{d}\mu^X(s)$ is the disintegration of μ by vertical slices. A similar result holds for the Berger measure η_i of the associated i-th vertical weighted shift $W_{\beta^{(i)}}$ ($i \geqslant 0$) for $W_{(\alpha,\beta)}$. (For a measure μ on $X \times Y$, $\mathrm{d}\mu^X(s) := \int_Y \mathrm{d}\mu(s,t)$ and $\mathrm{d}\mu^Y(t) := \int_X \mathrm{d}\mu(s,t)$.)*

(v) (*See* [5].) *Assume that $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ and $W_{(\alpha,\beta)}|_{\mathcal{M}_1}$ is subnormal with associated measure $\mu_{\mathcal{M}_1}$. Then $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty \Leftrightarrow \{\frac{1}{t} \in L^1(\mu_{\mathcal{M}_1})$, $\beta_{00}^2 \leqslant (\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}_1})})^{-1}$ and $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}_1})} (\mu_{\mathcal{M}_1})_{ext}^X \leqslant \xi_0\}$. (For a probability measure ν, $(\nu)_{ext}$ denotes the extremal measure associated with ν, defined as $\mathrm{d}(\nu)_{ext}(s,t) := (1-\delta_0(t)) \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}_1})} \frac{1}{t}\,\mathrm{d}\mu(s,t)$.) In the case when $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty$, the Berger measure μ of $W_{(\alpha,\beta)}$ is given by $\mu = \beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}_1})} (\mu_{\mathcal{M}_1})_{ext} + (\xi_0 - \beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}_1})} (\mu_{\mathcal{M}_1})_{ext}^X) \times \delta_0$.*

A probability measure ϵ on $X \times X$ is said to be *diagonal* if $\operatorname{supp}\epsilon \subseteq \{(s,s)\colon s \in X\}$. We now have:

Theorem 2.2. (a) *Let W_ω be a unilateral weighted shift, let $\Theta(\omega)$ be the canonical embedding of ω, and let $k \geqslant 1$. Then W_ω is k-hyponormal if and only if $\Theta(W_\omega) \in \mathfrak{C}_k$.* (b) *$W_\omega$ is subnormal if and only if $\Theta(W_\omega) \in \mathfrak{H}_\infty$; in this case, the Berger measure ϵ of $\Theta(W_\omega)$ is diagonal. Moreover, ϵ^X is the Berger measure of W_ω.*

Sketch of proof. First recall that, by Lemma 2.1(i) and (iii), W_ω is k-hyponormal if and only if $H(k;n) \geqslant 0$ (all $n \geqslant 0$), and $\Theta(W_\omega) \in \mathfrak{C}_k \iff M_{\mathbf{u}}(k) \geqslant 0$ (all $\mathbf{u} \equiv (u_1,u_2) \in \mathbb{Z}_+^2$). Next, looking at Fig. 1(i), it is easy to conclude that the moments of $\Theta(W_\omega)$, $\gamma_{\mathbf{u}}^{\Theta(W_\omega)}$, and the moments of W_ω, $\gamma_k^{W_\omega}$, are related by the identity $\gamma_{\mathbf{u}}^{\Theta(W_\omega)} = \gamma_{u_1+u_2}^{W_\omega}$. Using straightforward row and column operations, it follows at once that $H(k;n) \geqslant 0$ (all $n \geqslant 0$) if and only if $M_{\mathbf{u}}(k) \geqslant 0$ (all $\mathbf{u} \in \mathbb{Z}_+^2$). By Lemma 2.1(i) and (iii), $W_\omega \in \mathfrak{H}_k$ if and only if $\Theta(W_\omega) \in \mathfrak{H}_k$. This establishes (a).

To prove the first part of (b), recall the Bram–Halmos Criterion (in one variable [1] and two variables [9]), that subnormality is equivalent to k-hyponormality for every $k \geqslant 1$. The remaining statements follow from direct computations using disintegration-of-measure techniques (cf. [1,6]) and Lemma 2.1(v). □

By Lemma 2.1(ii) and Theorem 2.2, we have:

Corollary 2.3. (i) $\mathfrak{C}_\infty \subsetneq \cdots \subsetneq \mathfrak{C}_k \subsetneq \cdots \subsetneq \mathfrak{C}_1 \subsetneq \mathfrak{C}_0$. (ii) *For positive real numbers* $(x_i)_{i=1}^n$ *and* $(\omega_i)_{i=0}^k$, *if* $W_\omega \equiv \operatorname{shift}(x_n, \dots, x_1, (\omega_0, \dots, \omega_k)^\wedge)$, *then* W_ω *is subnormal if and only if either* $\Theta(W_\omega) \in \mathfrak{C}_{([\frac{k+1}{2}]+1)}$ $(n=1)$ *or* $\Theta(W_\omega) \in \mathfrak{C}_{([\frac{k+1}{2}]+2)}$ $(n>1)$.

Using disintegration-of-measure techniques [6] and Lemma 2.1(v), we have:

Theorem 2.4. *Let* ϵ *be a diagonal probability measure on* $X \times Y$. *Then we have* $\epsilon^X = \epsilon^Y$ *and for all* $f \in C(X \times Y)$,

$$\iint f(s,t)\,\mathrm{d}\epsilon(s,t) = \int_X \left(\int_Y f(s,t)\,\mathrm{d}\Phi_t(t) \right) \mathrm{d}\epsilon^X(s) = \int_X \left(\int_Y f(s,t)\,\mathrm{d}\epsilon^Y(t) \right) \mathrm{d}\epsilon^X(s)$$
$$= \int_Y \left(\int_X f(s,t)\,\mathrm{d}\epsilon^X(s) \right) \mathrm{d}\epsilon^Y(t).$$

Corollary 2.5. *Let* $\Theta(W_\omega)$ *be subnormal with Berger measure* ϵ. *Then for* $m_1, m_2 \geqslant 1$, $\Theta(W_\omega)^{(m_1,m_2)} \in \bigoplus \mathfrak{H}_\infty$. *Furthermore, the Berger measure of* $\Theta(W_\omega)^{(m_1,m_2)}|_{\mathcal{H}_{(0,0)}^{(m_1,m_2)}}$ *is* $\mathrm{d}\epsilon_{(0,0)}^{(m_1,m_2)}(s,t) = \mathrm{d}\epsilon(s^{\frac{1}{m_1}}, t^{\frac{1}{m_2}})$.

Fig. 1(ii) shows the general form of a pair $W_{(\alpha,\beta)}$ in $\mathcal{DC}$, and that it is uniquely determined by the four parameters σ, τ, a and ϵ, where ϵ is the Berger measure of $c(W_{(\alpha,\beta)})$. Thus, in what follows we will identify a pair $W_{(\alpha,\beta)} \in \mathcal{DC}$ with the 4-tuple $\langle \sigma, \tau, a, \epsilon \rangle$. We now let $\psi := (\tau)_1 - a^2 \|\frac{1}{s}\|_{L^1(\epsilon^X)} \epsilon^Y$ and $\varphi := \sigma - \beta_{(0,0)}^2 \|\frac{1}{t}\|_{L^1(\psi)} \delta_0 - a^2 \beta_{(0,0)}^2 \|\frac{1}{t}\|_{L^1(\epsilon^Y)} \frac{\epsilon^X}{s}$, where $(\tau)_1$ is the Berger measure of the subnormal $\operatorname{shift}(y_1, y_2, \dots)$. Clearly ψ and φ are measures. When $W_{(\alpha,\beta)} \in \mathcal{DC}$, we establish that

$$W_{(\alpha,\beta)}^{(m_1,m_2)} = \bigoplus_{p=0}^{m_1-1} \bigoplus_{q=0}^{m_2-1} \langle \sigma_{(p,q)}^{(m_1,m_2)}, \tau_{(p,q)}^{(m_1,m_2)}, a_{(p,q)}^{(m_1,m_2)}, \epsilon_{(p,q)}^{(m_1,m_2)} \rangle,$$

where $\langle \sigma_{(p,q)}^{(m_1,m_2)}, \tau_{(p,q)}^{(m_1,m_2)}, a_{(p,q)}^{(m_1,m_2)}, \epsilon_{(p,q)}^{(m_1,m_2)} \rangle$ is the 4-tuple associated to the restriction of $W_{(\alpha,\beta)}^{(m_1,m_2)}$ to the reducing subspace $\mathcal{H}_{(p,q)}^{(m_1,m_2)}$. We note that $W_{(\alpha,\beta)}^{(m_1,m_2)} = (W_{(\alpha,\beta)}^{(m_1,1)})^{(1,m_2)}$.

3. Main results

Theorem 3.1. *Let* $W_{(\alpha,\beta)} \in \mathcal{DC}$. *Then for* $q = 0, 1$, $W_{(\alpha,\beta)}^{(m_1,m_2)}|_{\mathcal{H}_{(0,q)}^{(m_1,m_2)}} \in \mathfrak{H}_\infty$ *if and only if* $\psi_{(0,q)}^{(m_1,m_2)}$ *and* $\varphi_{(0,q)}^{(m_1,m_2)}$ *are positive measures. In the case when* $W_{(\alpha,\beta)}^{(m_1,m_2)}|_{\mathcal{H}_{(0,q)}^{(m_1,m_2)}} \in \mathfrak{H}_\infty$, *the Berger measure* μ *of* $W_{(\alpha,\beta)}^{(m_1,m_2)}|_{\mathcal{H}_{(0,q)}^{(m_1,m_2)}}$ *is given by*

$$\mu = \varphi_{(0,q)}^{(m_1,m_2)} \times \delta_0 + \left(\beta_{(0,q)}^{(m_1,m_2)}\right)^2 \left(\frac{(a_{(0,q)}^{(m_1,m_2)})^2 \epsilon_{(0,q)}^{(m_1,m_2)}}{st} + \delta_0 \times \frac{\psi_{(0,q)}^{(m_1,m_2)}}{t} \right).$$

Theorem 3.2. *Let* $W_{(\alpha,\beta)} \in \mathcal{DC}$ *and let* $(m_1, m_2) \geqslant (1,1)$. *Then* $W_{(\alpha,\beta)}^{(m_1,m_2)} \in \mathfrak{H}_\infty$ *if and only if* $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty$.

Sketch of proof. It suffices to prove necessity, which we do in three steps.

(a) Without loss of generality, we can always assume $m_1 = 1$. Although the class $\mathcal{DC}$ is not invariant under powers, using disintegration-of-measure techniques for multivariable weighted shifts [6], we can show that the pair (φ, ψ) associated with $W_{(\alpha,\beta)}$ is directly related to the pairs $(\varphi_{(0,q)}^{(1,m_2)}, \psi_{(0,q)}^{(1,m_2)})$ associated with the direct summands in the orthogonal decomposition of $W_{(\alpha,\beta)}^{(1,m_2)}|_{\mathcal{H}_{(0,q)}^{(m_1,m_2)}}$.

(b) If a power $W_{(\alpha,\beta)}^{(1,m_2)}|_{\mathcal{H}_{(0,q)}^{(m_1,m_2)}}$ is subnormal, the functionals $\varphi_{(0,q)}^{(1,m_2)}$ and $\psi_{(0,q)}^{(1,m_2)}$ are both positive measures.

(c) It then follows that φ and ψ are positive measures, and therefore $W_{(\alpha,\beta)}$ is subnormal. □

We conclude this section by showing that the k-hyponormality of $\Theta(W_\omega)$ $(k = 1, 2)$ is not invariant under the action $(m_1, m_2) \mapsto \Theta(W_\omega)^{(m_1,m_2)}$ $(m_1, m_2 \geqslant 1)$.

Example 3.3. (Please refer to the notation introduced in Lemma 2.1(ii).) For $\sqrt{2} \leqslant t \leqslant \frac{3}{2}$, let $W_\omega = \text{shift}(x_5, x_4, x_3, x_2, x_1, (\omega_0, \omega_1, \omega_2)^\wedge)$, where $x_4 := \sqrt{\frac{1}{5}}$, $x_3 := \frac{1}{2}$, $x_2 := \sqrt{\frac{1}{2}}$, $x_1 := 1$, $\omega_0 := \sqrt{t}$, $\omega_1 := \sqrt{t+1}$ and $\omega_2 := \sqrt{t+2}$. Then we have (i) $\Theta(W_\omega) \in \mathfrak{C}_1 \iff 0 < x_5 \leqslant \sqrt{\frac{1}{5}}$; (ii) $\Theta(W_\omega)^{(2,1)} \in \mathfrak{C}_1 \iff 0 < x_5 \leqslant \sqrt{\frac{25}{128}}$; (iii) $\Theta(W_\omega) \in \mathfrak{C}_2 \iff 0 < x_5 \leqslant \sqrt{\frac{5}{26}}$; (iv) $\Theta(W_\omega)^{(2,1)} \notin \mathfrak{C}_2$; (v) $\Theta(W_\omega) \notin \mathfrak{C}_3$.

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