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Number Theory

An algorithm computing non-solvable spectral radii of p-adic differential equations

Un algorithme pour le calcul des rayons de convergence non solubles des équations différentielles p-adiques

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A B S T R A C T

We obtain an algorithm computing explicitly the values of the non-solvable spectral radii of convergence of the solutions of a differential module over a point of type 2, 3 or 4 of the Berkovich affine line.

R É S U M É

Nous obtenons un algorithme pour le calcul explicite des valeurs des rayons de convergence spectrales non solubles des solutions d'un module différentiel sur un point de type 2, 3 ou 4 de la droite affine de Berkovich.

0. Introduction

By a theorem of Young [10], the small values of the radii of convergence of the solutions of a differential operator are explicit, and coincide with the small slopes of the Newton polygon of a differential operator attached to the module (cf. Proposition 2.2). Larger radii are not immediately readable on the coefficients of the operator. This discrepancy is a peculiarity of the p-adic world: only small radii are "*visible*". To overcome this problem, B. Dwork observed that the pull-back by the Frobenius functor increases the radii of the solutions, and together with G. Christol [3] they constructed an inverse of the Frobenius functor (often called Frobenius antecedent) in order to make the radii of the solutions smaller and hence explicitly intelligible in a cyclic basis. Although theoretically satisfactory, the inversion of Frobenius is a completely implicit operation. Moreover, the antecedent exists only if all the radii of the solutions are not small. So one is obliged to factorize the module by the radii of the solutions if one wants to understand the non-minimal radii of the solutions. The factorization is also an implicit operation. Recently, in [6], K. Kedlaya observed that the Frobenius push-forward operation has essentially the same effects as the inversion of the Frobenius on the radii of the solutions, and he is able to control the exact behavior of all the radii of the solutions under this operation (even small radii).[1] The Frobenius push-forward functor is completely explicit, and it allows us to obtain a concrete algorithm to compute the radii of the solutions that are not maximal (i.e. non-solvable). The price to pay is that the dimension of the push-forward by Frobenius is p-times that of the original module,

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[1] The very first reference for this is [1], in which G. Christol introduces the push-forward, and its relation with the pull-back, and uses it to prove the inversion of Frobenius (existence of the antecedent) under the condition that the solutions of the module are bounded.

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so that the complexity of the algorithm is multiplied by p at each application of the push-forward.[2] Hence, more the radius is large, more complexity increases. Moreover, the algorithm admits an end if and only if all the radii are not maximal (i.e. non-*solvable*). Eventually we provide the algorithm, but we avoid to provide a complete formula as one does for rank-one equations (cf. [2][3]). Indeed, complexity seems so great that the formula would result not being useful to be written. Explicit examples are quite complicated, even in the rank-one case. This note is intended to make explicit the computations, and the link between the different results, with a view to making them explicitly calculable by a computer.

1. Radii of convergence

Let $(K,|.|)$ be a complete field with respect to an ultrametric absolute value $|.|: K \to \mathbb{R}_{\geqslant 0}$. Let L/K be a complete valued field extension, let $c \in L$, and $\rho > 0$. For all polynomial $P(T) := \sum_i a_i T^i \in K[T]$ define $|P|_{c,\rho} := \sup_{n\geqslant 0} |P^{(i)}(c)|\rho^n/n!$. The setting $|P_1/P_2|_{c,\rho} := |P_1|_{c,\rho}/|P_2|_{c,\rho}$ defines an absolute value on the field of fraction $K(T) := \mathrm{Frac}(K[T])$, and hence a Berkovich point $\xi_{c,\rho}$ of the affine line $\mathbb{A}^{1,\mathrm{an}}_K$. Since $\rho > 0$ if $c \in K$ one obtains in this way all the points of type 2 or 3 of $\mathbb{A}^{1,\mathrm{an}}_K$, if one allows $c \notin K$ one also has all points of type 4. The derivative $\mathrm{d}/\mathrm{d}T$ extends by continuity to the completion $\mathscr{H}_{c,\rho}$ of $(K(T),|.|_{c,\rho})$. A differential module over $\mathscr{H}_{c,\rho}$ is a finite dimensional $\mathscr{H}_{c,\rho}$-vector space M together with a K-linear map $\nabla: \mathrm{M} \to \mathrm{M}$ satisfying $\nabla(fm) = d(f)m + f\nabla(m)$, for all $f \in \mathscr{H}_{c,\rho}$, $m \in \mathrm{M}$. Let $r(\xi_{c,\rho}) \geqslant \rho$ be the radius of the point $\xi_{c,\rho}$ (cf. [9, Section 1.3.1]). If Ω/K is a complete valued field extension such that $\mathbb{A}^{1,\mathrm{an}}_\Omega$ has an Ω-rational point $t_{c,\rho} \in \Omega$ lifting $\xi_{c,\rho}$, then $r(\xi_{c,\rho})$ is the radius of the largest open disk $\mathrm{D}^-_\Omega(t_{c,\rho}, r(\xi_{c,\rho}))$ satisfying $\mathrm{D}^-_\Omega(t_{c,\rho}, r(\xi_{c,\rho})) \cap K^{\mathrm{alg}} = \emptyset$.

Definition 1.1. Let $r := \mathrm{rank}(\mathrm{M})$. For $i = 1,\dots,r$ we denote by $\mathcal{R}_i = \mathcal{R}_i^{\mathrm{M,sp}}(\xi_{c,\rho}) \leqslant r(\xi_{c,\rho})$ the radius of the largest open disk in $\mathrm{D}^-_\Omega(t_{c,\rho}, r(\xi_{c,\rho}))$ centered at $t_{c,\rho}$ on which M has at least $r - i + 1$ Ω-linearly independent Taylor solutions (cf. [9, Section 4.2] or [6, 11.9]). One has $\mathcal{R}_1 \leqslant \mathcal{R}_2 \leqslant \cdots \leqslant \mathcal{R}_r$.

We say that $\mathcal{R}_i$ is *solvable* if $\mathcal{R}_i = r(\xi_{c,\rho})$. In this paper, we provide an algorithm computing non-solvable radii.

2. Comparison of Newton polygons and computation of small radii

Let $r \geqslant 1$ be an integer. A *slope sequence* is the data of r real numbers $s_1 \leqslant \cdots \leqslant s_r$ in increasing order. Define the i-th partial height as $h_i := s_1 + \cdots + s_i$. A slope sequence defines univocally a convex function $h: [0,r] \to \mathbb{R}$ by $h(0) := 0$, $h(i) := h_i$, and $h(x) = s_i x + (h_i - i \cdot s_i)$ for all $x \in]i-1,i]$, $i = 0,\dots,r$. The function h is called the *Newton polygon with slopes* $s_1 \leqslant \cdots \leqslant s_r$.

Definition 2.1. The Newton polygon with slopes $s_i := s_i^{\mathrm{M,sp}}(\xi_{c,\rho}) := \ln(\mathcal{R}_i^{\mathrm{M,sp}}(\xi_{c,\rho}))$ is called the *spectral Newton polygon* of M. We denote by $h_i := h_i^{\mathrm{M,sp}}(\xi_{c,\rho})$ its i-th partial height.

Let $\mathcal{L} = \sum_{i=0}^r g_{r-i} d^i$, $g_i \in \mathscr{H}_{c,\rho}$, be a differential operator with $g_0 = 1$. Let $v_0 = 0$, and for all $i = 1,\dots,r$ let $v_i := -\ln(|g_i|_{c,\rho}/\omega^i)$, where $\omega := \lim_n |n!|^{1/n}$. It is understood that if $g_i = 0$ then $v_i = +\infty$. Define the *spectral Newton polygon* $NP(\mathcal{L})$ as the intersection of all upper half planes $H_{a,b} := \{(x,y) \in \mathbb{R}^2$ such that $y \geqslant ax + b\}$ with $\{(i,v_i)\}_{i=0,\dots,r} \subset H_{a,b}$. Let $h^{\mathcal{L}}: [0,r] \to \mathbb{R}$ be the convex function whose epigraph is $NP(\mathcal{L})$: $h^{\mathcal{L}}(x) = \min\{y$ such that $(x,y) \in NP(\mathcal{L})\}$. Explicitly, one has $h_i^{\mathcal{L}} := h^{\mathcal{L}}(i) = \sup_{s\in\mathbb{R}}\{s \cdot i + \min_{j=0,\dots,r} v_j - s \cdot j\}$. Then $NP(\mathcal{L})$ is the Newton polygon with slopes $\{s_i^{\mathcal{L}} := h_i^{\mathcal{L}} - h_{i-1}^{\mathcal{L}}\}_{i=1,\dots,r}$.

Proposition 2.2. *(See [10].) Let $\mathcal{L}$ be a differential operator as above and let* M *be the differential module defined by $\mathcal{L}$. Let $C := \ln(\omega \cdot r(\xi_{c,\rho}))$, then for all $i = 1,\dots,r$, one has*

$$\min\big(s_i^{\mathrm{M,sp}}, C\big) = \min\big(s_i^{\mathcal{L}}, C\big). \tag{2.1}$$

Remark 2.3. In order to apply (2.1), we need an algorithm to find a cyclic basis of M (cf. Section 3). If the absolute value of K is trivial on $\mathbb{Z}$ (i.e. if $|n| = 1$ for all $n \in \mathbb{Z} - \{0\}$), then $\omega = 1$, and Proposition 2.2 allows us to find all the radii $\mathcal{R}_i$. If the absolute value of K is p-adic (i.e. if $|p| < 1$), then $\omega = |p|^{\frac{1}{p-1}} < 1$. In this case, we also need a technique (Frobenius push-forward) making the (non-solvable) radii smaller than $\omega r(\xi_{c,\rho})$ (cf. Section 4).

[2] Xavier Caruso recently pointed out that the explicit factorization of an operator by the radii of convergence seems to be concretely implementable into a machine. This would highly reduce the complexity of the present algorithm, since by considering the right factor of the push-forward by Frobenius the dimension remains constant at each step.

[3] The formula that we have contributed to prove in [2] is based on a completely different approach, and it uses Witt vectors (following techniques of [8]) to explicitly describe Taylor solutions of a rank-one differential equation.

3. Explicit cyclic vector

Let (F,d) be a differential field and let $F\langle d\rangle = \bigoplus_{i\geqslant 0} F\circ d^i$ be the Weil algebra of differential operators. The multiplication of $F\langle d\rangle$ extends that of $F = F\circ d^0$ by the rule $d\circ f = f\circ d + d(f)$, for all $f\in F$. Finite dimensional differential modules over F are exactly torsion left $F\langle d\rangle$-modules. The so-called *cyclic vector theorem* asserts that all differential modules are not only torsion modules over $F\langle d\rangle$: they are cyclic modules, i.e. of the form $(\mathrm{M},\nabla) = (F\langle d\rangle/F\langle d\rangle\mathcal{L}, d)$, for some $\mathcal{L} := \sum_{i=0}^{r} g_{r-i}d^i \in F\langle d\rangle$, with $g_0 = 1$, $g_i\in F$. The image in the quotient of $\{1, d, d^2, \dots, d^{r-1}\}$ forms a basis of M, and the action of ∇ is given by the multiplication by d in the quotient. In fact, the cyclic vector theorem is equivalent to the existence of an element $c\in \mathrm{M}$, called cyclic vector, such that $\{c, \nabla(c), \nabla^2(c), \dots, \nabla^{r-1}(c)\}$ is a basis of M. In this case, if $c_i := \nabla^i(c)$, and if $\nabla^r(c_0) = \sum_{i=0}^{r-1} f_i c_i$, then $f_i = -g_{r-i}$. The existence of such a vector is due to [4, Ch. II, Lemme 1.3]. Subsequently, N.M. Katz provided the following explicit algorithm:

Theorem 3.1. *(See [5].) Let (M,∇) be a differential module over (F,d) of rank r, and let $\mathbf{e} := \{e_0,\dots,e_{r-1}\}\subset \mathrm{M}$ be a basis of* M. *Let $a_0,\dots,a_{r(r-1)}\in F$ be $r(r-1)+1$ distinct constants i.e. $d(a_i) = 0$. Then at least one of the following elements of* M *is a cyclic vector*:

$$c(\mathbf{e}, T-a_i) := \sum_{j=0}^{r-1} \frac{(T-a_i)^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \nabla^k(e_{j-k}). \tag{3.1}$$

Remark 3.2. The explicit base change matrices to the Katz's cyclic basis are quite complicated and have been explicitly computed in [7]. On the other hand, the proof of the existence of a cyclic vector of Deligne [4, Ch. II, Lemme 1.3] proves that the family of cyclic vectors in a given module is the complement of a hypersurface. The base change matrices of Katz's algorithm are quite involved and hard to find by hand, even in small examples. It is often convenient to pick an arbitrary vector and test if it is cyclic.

Remark 3.3. One shall avoid the use of a cyclic basis using [6, Lemma 6.7.3, Theorem 6.7.4, Conjecture 4.4.9]. This should permit to compute "small" radii of Proposition 2.2 directly in terms of the norms of the eigenvalues of the characteristic polynomial of the original matrix of ∇, without using any cyclic vector.

4. Frobenius push-forward and explicit computation of larger radii

In this section, we assume that $|p| < 1$ (cf. Remark 2.3).

Hypothesis 4.1. $\mathcal{R}_i^{\mathrm{M,sp}}$ is insensitive to scalar extensions of K, and by translations. So in the sequel we will assume $c = 0$ and replace the indexation (c,ρ) by ρ. In this case, one has $r(\xi_\rho) = \rho$. We then work with $|.|_\rho$, ξ_ρ, $\mathscr{H}_\rho$, $r(\xi_\rho) = \rho$ with the evident meaning of notation. If the reader needs to preserve the setting (c,ρ), the same computations hold replacing the map $\varphi : T\mapsto T^p$ by $T\mapsto (T-c)^p + c$. Or alternatively one also can preserve $\varphi : T\to T^p$, and proceed as in [9, Section 7] to check the behavior of the radii by Frobenius at points that are close enough to the segment $\rho\mapsto |.|_{0,\rho}$ (this is often necessary if one needs the slopes of $\mathcal{R}_i^{\mathrm{M,sp}}$ along a Berkovich path $\rho\mapsto |.|_{c,\rho}$, with $c\in K$ and ρ close to $|c|$).

Let $\widetilde{T}$, T be two variables, and let $\varphi : K(T)\to K(\widetilde{T})$ be the ring morphism sending T into $\widetilde{T}^p$. This extends into an isometric inclusion $\varphi : \mathscr{H}_{\rho^p}\to \mathscr{H}_\rho$ of degree p. One has the rule $\frac{\mathrm{d}}{\mathrm{d}T}(f(T)) = \frac{\mathrm{d}/\mathrm{d}\widetilde{T}}{p\widetilde{T}^{p-1}}(f(T))$, for all $f\in K(T)$. We call

$$d_{\rho^p} := \frac{\mathrm{d}}{\mathrm{d}T}, \qquad d_\rho := \frac{\mathrm{d}}{\mathrm{d}\widetilde{T}}, \qquad \widetilde{d}_{\rho^p} := \bigl(p\widetilde{T}^{p-1}\bigr)^{-1}\frac{\mathrm{d}}{\mathrm{d}\widetilde{T}}. \tag{4.1}$$

Let $(\widetilde{\mathrm{M}},\nabla)$ be a differential module over $(\mathscr{H}_\rho, d_\rho)$ of rank r. Since $(\widetilde{d}_{\rho^p})_{|\mathscr{H}_{\rho^p}} = d_{\rho^p}$, then $(\widetilde{\mathrm{M}}, (p\widetilde{T}^{p-1})^{-1}\nabla)$ is a differential module over $(\mathscr{H}_\rho, \widetilde{d}_{\rho^p})$ that can be seen (by restriction of the scalars) as a differential module over $(\mathscr{H}_{\rho^p}, d_{\rho^p})$ of rank pr. We call $(\varphi_*\widetilde{\mathrm{M}}, \varphi_*\nabla)$ the differential module so obtained.

4.1. Explicit matrix of $\varphi_*\nabla$

One has a direct sum decomposition $\mathscr{H}_\rho = \bigoplus_{k=0}^{p-1}\varphi(\mathscr{H}_{\rho^p})\cdot\widetilde{T}^k$, so that each $g(\widetilde{T})\in\mathscr{H}_\rho$ can be uniquely written as $g(\widetilde{T}) = \sum_{k=0}^{p-1} g_k(\widetilde{T}^p)\widetilde{T}^k = \sum_{k=0}^{p-1} g_k(T)\widetilde{T}^k$. The derivation $\widetilde{d}_{\rho^p}$ stabilizes globally each factor and $\widetilde{d}_{\rho^p}(g_k(T)\widetilde{T}^k) = (d_{\rho^p}(g_k(T)) + \frac{k}{pT}g_k(T))\widetilde{T}^k$. For all $g(\widetilde{T})\in\mathscr{H}_\rho$, we define $\varphi_*(g)(T)\in M_{p\times p}(\mathscr{H}_{\rho^p})$ as the matrix of the multiplication by $g(\widetilde{T})/(p\widetilde{T}^{p-1})$, with respect to the basis $1, \widetilde{T}, \dots, \widetilde{T}^{p-1}$ over $\mathscr{H}_{\rho^p}$. One has

$$\varphi_*(g)(T) = (pT)^{-1} \cdot \begin{pmatrix} Tg_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & \cdots & \cdots & Tg_0(T) \\ g_0(T) & Tg_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & \cdots & Tg_1(T) \\ g_1(T) & g_0(T) & Tg_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & Tg_2(T) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{p-2}(T) & g_{p-3}(T) & \cdots & \cdots & g_1(T) & g_0(T) & Tg_{p-1}(T) \end{pmatrix}. \tag{4.2}$$

Notice that the terms over the diagonal are multiplied by T. Let $(\widetilde{\mathrm{M}}, \widetilde{\nabla})$ be a differential module over $\mathscr{H}_\rho$. Fix an $\mathscr{H}_\rho$-linear isomorphism $\mathscr{H}_\rho^r \xrightarrow{\sim} \widetilde{\mathrm{M}}$ (i.e. a basis of $\widetilde{\mathrm{M}}$), and let $\frac{d}{d\widetilde{T}} - G(\widetilde{T})$ be the map $\widetilde{\nabla}$ in this basis, where $G(\widetilde{T}) = (g_{i,j}(\widetilde{T}))_{i,j=1,\dots,r} \in M_{r\times r}(\mathscr{H}_\rho)$. Writing $\mathscr{H}_\rho^r = (\bigoplus_{k=0}^{p-1} \varphi(\mathscr{H}_{\rho^p}) \cdot \widetilde{T}^k)^r$, one sees that the multiplication by $(p\widetilde{T}^{p-1})^{-1}G(\widetilde{T})$ is given by the block matrix:

$$\varphi_*(G)(T) := \big(\varphi_*(g_{i,j})(T)\big)_{i,j=1,\dots,r} \in M_{pr\times pr}(\mathscr{H}_{\rho^p}). \tag{4.3}$$

The action of $\varphi_*(\widetilde{\nabla})$ is then given by $d_{\rho^p} + N_r - \varphi_*(G)(T)$, where $N_r \in M_{pr\times pr}(\mathbb{N})$ is a diagonal matrix whose i-th entry of the diagonal is $(q(i)-1)/pT$, where $0 \leqslant q(i) < p$ is the rest of the Euclidean division of i by p: $i = s\cdot p^n + q(i)$. N_r then has p blocks on the diagonal of the form $(pT)^{-1}\mathrm{diag}(0, 1, 2, \dots, p-1)$.

4.2. Behavior of the radii by Frobenius push-forward

Theorem 4.2. *(See [6, Theorem 10.5.1].) Let $\mathcal{R}_1 \leqslant \cdots \leqslant \mathcal{R}_r$ be the spectral radii of $\widetilde{\mathrm{M}}$ at ξ_ρ (cf. Definition 1.1). Let i_1 be such that $\mathcal{R}_{i_1} \leqslant \omega\rho < \mathcal{R}_{i_1+1}$.[4] Then, up to permutation, the spectral radii of $\varphi_*\widetilde{\mathrm{M}}$ at ξ_{ρ^p} are:*

$$\bigcup_{i\leqslant i_1} \{\underbrace{|p|\rho^{p-1}\mathcal{R}_i, \dots, |p|\rho^{p-1}\mathcal{R}_i}_{p\text{-times}}\} \bigcup_{i>i_1} \{\mathcal{R}_i^p, \underbrace{\omega^p\rho^p, \dots, \omega^p\rho^p}_{p-1\text{-times}}\}. \tag{4.4}$$

If $s_1 \leqslant \cdots \leqslant s_r$ is the slope sequence of the spectral Newton polygon of $\widetilde{\mathrm{M}}$ at ξ_ρ, and if $i_0 \geqslant i_1$ satisfies $\mathcal{R}_{i_0} < \rho = \mathcal{R}_{i_0+1}$,[5] then by Theorem 4.2 the slope sequence associated with $\varphi_*\widetilde{\mathrm{M}}$ at ξ_{ρ^p} is:

$$\overbrace{\ln(|p|\rho^{p-1}) + s_1 = \cdots = \ln(|p|\rho^{p-1}) + s_1}^{p\text{-times}} \leqslant \cdots \leqslant \overbrace{\ln(|p|\rho^{p-1}) + s_{i_1} = \cdots = \ln(|p|\rho^{p-1}) + s_{i_1}}^{p\text{-times}}$$
$$\leqslant \overbrace{\ln(\omega^p\rho^p) = \cdots = \ln(\omega^p\rho^p)}^{(p-1)(r-i_1)\text{-times}} < ps_{i_1+1} \leqslant \cdots \leqslant ps_{i_0} < \overbrace{\ln(\rho^p) = \cdots = \ln(\rho^p)}^{(r-i_0)\text{-times}}. \tag{4.5}$$

We have two main goals here. Firstly the sequence $s_1 \leqslant \cdots \leqslant s_r$ is perfectly determined by the knowledge of the slope sequence (4.5) of $\varphi_*\widetilde{\mathrm{M}}$ (even if some of the s_i are equal to the critical value $\ln(\omega\rho)$), see [9, Proposition 6.17] for a more precise statement. Secondly the values of s_i satisfying $\ln(\omega^{1/p}\rho) \leqslant s_i < \ln(\omega^{1/p}\rho)$ correspond to small radii[6] of $\varphi_*\widetilde{\mathrm{M}}$ that are explicitly intelligible by Proposition 2.2. Iterating this construction by performing several times the push-forward, one obtains an explicit algorithm that computes all the non-solvable radii $\mathcal{R}_1, \dots, \mathcal{R}_{i_0}$ in a finite number of steps. Once this has been achieved, one knows in fact all the spectral radii since the remaining radii are all equal to ρ. Unfortunately, Proposition 2.2 does not furnish any information about radii that are larger than $\omega\rho$, so (unless the radii are all not solvable) it seems impossible to know whether the algorithm is ended or if one needs more applications of the Frobenius push-forward.

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[4] It is understood that $i_1 = 0$ if $\omega\rho < \mathcal{R}_1$.
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