Partial Differential Equations

A Carleman estimate for the two dimensional heat equation with mixed boundary conditions

Inégalité de Carleman pour l’équation de la chaleur avec conditions mixtes en dimension deux

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1. Introduction

Let $\Omega$ be a bounded open connected set of $\mathbb{R}^2$ with $C^2$ boundary $\Gamma = \partial \Omega$. Let $\Gamma_D$ and $\Gamma_N$ be two subsets of $\Gamma$ such that: $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $\Gamma_D \cap \Gamma_N = \{S_1, S_2\}$. Let $\omega \subset \Omega$ be a non-empty open subset. For $T > 0$, we set $Q_T = \Omega \times (0,T)$, $\Sigma_{DT} = \Gamma_D \times (0,T)$, $\Sigma_{NT} = \Gamma_N \times (0,T)$ and $\Sigma_T = \Gamma \times (0,T)$. We will denote by $\nu(x)$ the outward unit normal to $\Omega$ at $x \in \Gamma$. We consider the following mixed boundary value problem:

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u & = 0 & \text{on } \Gamma_D, \\
\frac{\partial u}{\partial \nu} & = 0 & \text{on } \Gamma_N, \\
\end{cases}
\]

(1)

where $f$ is given in $L^2(\Omega)$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative of $u$. It is well known, see [8], that the solution of (1) is not in $H^2(\Omega)$, and more precisely the solution, according to [7] is given by:

\[
u(r, \theta) = u_r(r, \theta) + C_1 \sqrt{r_1} \sin \frac{\theta_1}{2} \chi_1 + C_2 \sqrt{r_2} \cos \frac{\theta_2}{2} \chi_2,
\]

(2)

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where \( u_R \in H^2(\Omega) \) is the regular part, \((r_j, \theta_j)\) are the local polar coordinates at \( S_j \), \( C_j \) are real constants, \( \chi_j \) are cut-off functions such that \( 0 \leq \chi_j \leq 1 \) and \( \chi_j = 1 \) on a neighborhood of \( S_j \).

The aim of this note is to establish a Carleman estimate for the following problem:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= f & \text{in} & Q_T, \\
u &= 0 & \text{on} & \Sigma_{DT}, \\
\frac{\partial u}{\partial v} &= 0 & \text{on} & \Sigma_{NT}, \\
u(\cdot, 0) &= u_0 & \text{in} & \Omega,
\end{aligned}
\]

(3)

where \( u_0 \in L^2(\Omega) \) and \( f \in L^2(Q_T) \).

Carleman estimates have many applications varying from the quantification of the unique continuation problems, inverse problems to stabilization and control theory. These applications are the motivation to prove a suitable Carleman estimate for our problem. To the best of our knowledge, very few results on Carleman estimates in the presence of singularities have been established. We cite [2] for the Laplace equation for a domain with a corner, [1] for the heat equation in a singular domain and [3] for the wave equation with mixed conditions using microlocal approach. Our methodology here is in a similar spirit of [1,4–6].

In order to get well-posedness for (3), we define the following spaces:

\[ V = \{ u \in H^1(\Omega); \ u = 0 \text{ on } \Gamma_0 \} \]

and

\[
D(-\Delta) = \left\{ u \in V; \ \Delta u \in L^2(\Omega); \ \frac{\partial u}{\partial v} = 0 \text{ on } \Gamma_N \right\},
\]

\[
= \left\{ u \in V \cap H^2(\Omega); \ \frac{\partial u}{\partial v} = 0 \text{ on } \Gamma_N \right\} \oplus \text{span} \left\{ t^2 \sin \frac{\theta}{2}, t^2 \cos \frac{\theta}{2} \right\}.
\]

Problem (3) has a unique solution \( u \in C([0, T]; L^2(\Omega)) \cap C((0, T); D(-\Delta)) \cap C^1((0, T); L^2(\Omega)). \) Note that, even for very smooth data \( f \) and \( u_0 \), the solution of (3) is not regular near \( S_1 \) and \( S_2 \).

2. Main result

In the following, for \( k = 0, 1 \), we set:

\[
\xi_k(x, t) = \frac{e^{-(1)^k_{\beta(x)}}}{t(T-t)}, \quad \alpha_k(x, t) = \frac{e^{2\lambda_\beta t} - e^{-(1)^k_{\beta(x)}}}{t(T-t)}.
\]

Here, \( \lambda_\beta > 1 \) is a parameter and \( \beta(\chi(x)) \) is a function satisfying:

\[
\beta \in C^2(\Omega), \quad \beta(x) > 0 \ \text{in} \ \Omega, \quad \beta(x) = 0 \ \text{on} \ \partial \Omega, \quad |\nabla \beta| > 0 \ \text{on} \ \overline{\Omega} \setminus \omega',
\]

(5)

where \( \omega' \subset \omega \) is a non-empty open set. The existence of \( \beta \) satisfying (5) is proved in [6].

We state our main result, for \( \alpha = \alpha_0 \) and \( \xi = \xi_0 \):

**Theorem 2.1.** Given \( f \in L^2(Q_T) \) and \( u_0 \in L^2(\Omega) \). There exist \( s_0, \lambda_0 \) and \( C = C(\Omega, \omega) \) such that for any \( s > s_0, \lambda > \lambda_0 \) the solution of (3) satisfies:

\[
I(u, \xi_0, \alpha_0, Q_T) \leq C \left( \int_{Q_T} e^{-2s_0|a_0^{(s)}| |f|^2} dx dt + s_0^3 \lambda^4 \int_{\omega \times (0,T)} \xi_0^3 e^{2s_0^\alpha} |u|^2 dx dt \right),
\]

(6)

where

\[
I(u, \xi, \alpha, Q_T) = \int_{Q_T} e^{-2s_0 \langle (s\xi)^{-1} (|\partial_t u|^2 + |\Delta u|^2) + s\xi_0^2 \xi |\nabla u|^2 + s_0^2 \xi_0^3 |u|^2 \rangle} dx dt.
\]

(7)

**Sketch of the proof.** The proof will be given in four steps.

**Regularization of the solution:** The regularity of the solution \( u \) is not sufficient to do some integrations by parts. We then approximate \( u \) by a sequence of regular functions \( \{u_n\}_n \). Spaces \( D(-\Delta) \) and \( \overline{\text{D}(\Delta)} \) are dense in \( L^2(\Omega) \), then for \( u_0 \in L^2(\Omega) \) and \( f \in L^2(0, T; L^2(\Omega)) \), there exist sequences \( \{u_n\}_n \subset D(\Delta) \) and \( \{f_n\}_n \subset C^1(0, T; D(\Delta)) \) such that \( u_n \) converges to \( u_0 \) in \( L^2(\Omega) \) and \( f_n \) converges to \( f \) in \( L^2(0, T; L^2(\Omega)) \), then the problem (3) with \( f = f_n \) and \( u_0 = u_n \) has a unique solution \( u_n \in C^2((0, T); L^2(\Omega)) \cap C^1((0, T); D(\Delta)) \) and we can prove the following lemma:
Lemma 2.1. For $k = 0, 1$, set $\psi_n(x, t) = e^{-s_0\lambda_1(x, t)}u_0(x, t)$ and $\psi_0(x, t) = e^{-s_0\lambda_1(x, t)}u(x, t)$, we have

1. $(u_n)_n$ converges to $u$ in $L^2(0, T; V)$,
2. $(\Delta \psi_n, k)_n$ converges to $(\Delta \psi_k)$ in $L^2(Q_T)$.

In the sequel, and for simplicity, we will drop the index $n$.

**Approximation of the domain:** To remedy the lack of regularity of the solution near $S_1$ and $S_2$, we set, for $\varepsilon > 0$

$$
\Omega_\varepsilon = \Omega \setminus \bigcup_{k=1}^2 B(S_k, \varepsilon), \quad \partial \Omega_\varepsilon = \Gamma_{\varepsilon}^D \cup \Gamma_{\varepsilon}^N \cup C_\varepsilon^1 \cup C_\varepsilon^2.
$$

$$
Q_{\varepsilon, T} = \Omega_\varepsilon \times (0, T), \quad \Sigma_{\varepsilon, T} = \partial \Omega_\varepsilon \times (0, T), \quad C_{\varepsilon}^k = \partial B(S_k, \varepsilon) \cap \Omega,
$$

where $B(S_k, \varepsilon)$ is the ball of radius $\varepsilon$ and centred in $S_k$.

**Derivation of the Carleman estimate:** For $k = 0, 1$, let:

$$
\xi_k(x, t) = e^{(-1)^k\lambda_1(x)} \frac{t(T - t)}{t(T - t)}, \quad \alpha_k(x, t) = \frac{e^{2\lambda_1\beta(x)} - e^{(-1)^k\lambda_1(x)}}{t(T - t)},
$$

we set:

$$
\psi_k(x, t) = e^{-s\alpha(x, t)}u(x, t)
$$

and

$$
L\psi_k = M_1\psi_k + M_2\psi_k = F_k,
$$

where

$$
\begin{aligned}
M_1\psi_k &= 2s\lambda_2 \xi_k |\nabla \beta|^2 \psi_k + 2(-1)^k s\lambda_1 \nabla \psi_k \cdot \nabla \psi_k + \partial_t \psi_k, \\
M_2\psi_k &= -s^2 \lambda_2 |\nabla \beta|^2 \xi_k^2 \psi_k - \Delta \psi_k + s\lambda_1 \alpha_k \psi_k, \\
F_k &= e^{-s\alpha} f - (-1)^k s\lambda \xi_k \Delta \beta \psi_k + s\lambda_1^2 \xi_k^2 |\nabla \beta|^2 \psi_k.
\end{aligned}
$$

$(M_1\psi_k)_i$, $(M_2\psi_k)_j$ are respectively the $i$-th and the $j$-th term of $M_1\psi_k$ and of $M_2\psi_k$.

$$
\|M_1\psi_k\|_{L^2(Q_T)}^2 + \|M_2\psi_k\|_{L^2(Q_T)}^2 + 2 \sum_{i, j=1}^3 \langle (M_1\psi_k)_i, (M_2\psi_k)_j \rangle_{L^2(Q_T)} = \|F_k\|_{L^2(Q_T)}^2.
$$

Using integration by parts in (10), we derive the following inequality:

**Lemma 2.2.** There exist $s_0, \lambda_0$ and $C = C(\Omega, \omega)$ such that for any $s > s_0, \lambda > \lambda_0$,

$$
I(\psi_k, \xi_k, \alpha_k, Q_{\varepsilon, T}) + J(\psi_k, \xi_k, \alpha_k, \Sigma_{\varepsilon, T}) \leq C \left( \int_{Q_{\varepsilon, T}} e^{-2s\alpha_k} |f|^2 \, dx \, dt + s^3 \lambda^4 \int_{\omega \times (0, T)} \xi_k^3 |\psi_k|^2 \, dx \, dt \right).
$$

where $I(\psi_k, \xi_k, \alpha_k, \Sigma_{\varepsilon, T})$ is given by (7) and

$$
J(\psi_k, \xi_k, \alpha_k, \Sigma_{\varepsilon, T}) = 2(-1)^{k+1} \left( s^3 \lambda^3 \int_{\Sigma} \xi_k^3 |\nabla \beta|^2 \nabla \beta \cdot |\psi_k|^2 \, d\sigma \, dt + (-1)^k s\lambda_1^2 \int_{\Sigma} \xi_k \frac{\partial \psi_k}{\partial v} |\nabla \beta|^2 \psi_k \, d\sigma \, dt \\
+ 2s\lambda \int_{\Sigma} \xi_k (\nabla \beta, \nabla \psi_k) \frac{\partial \psi_k}{\partial v} \, d\sigma \, dt - s\lambda \int_{\Sigma} \xi_k |\nabla \psi_k|^2 (\nabla \beta, \nabla \psi_k) \, d\sigma \, dt \\
+ (-1)^k \int_{\Sigma} \frac{\partial \psi_k}{\partial v} \partial_t \psi_k \, d\sigma \, dt - s^2 \lambda_1 \int_{\Sigma} \xi_\alpha \nabla \beta \cdot |\psi_k|^2 \, d\sigma \, dt \right)
$$

and

$$
J(\cdot, \cdot, \Sigma_{\varepsilon, T}) = J(\cdot, \cdot, \Sigma_{\varepsilon, T}^D) + J(\cdot, \cdot, \Sigma_{\varepsilon, T}^N) + J(\cdot, \cdot, \Sigma_{\varepsilon, T}^1) + J(\cdot, \cdot, \Sigma_{\varepsilon, T}^2).
$$
Treatment of boundary terms and passing to the limit in $\varepsilon$: Since $\beta = 0$ on $\Gamma$ then $\alpha_0 = \alpha_1$, $\xi_0 = \xi_1$ and $\psi_0 = \psi_1$, we deduce that:

$$\psi_0 = \psi_1 = 0, \quad \frac{\partial \psi_0}{\partial v} = \frac{\partial \psi_1}{\partial v} \quad \text{on} \quad \Gamma_D^\varepsilon, \quad \frac{\partial \psi_0}{\partial v} = -\frac{\partial \psi_1}{\partial v}, \quad |\nabla \psi_0| = |\nabla \psi_1| \quad \text{on} \quad \Gamma_N^\varepsilon,$$

which implies that:

$$\sum_{k=0}^1 J(\psi_k, \xi_k, \alpha_k, \Sigma_D^\varepsilon) = \sum_{k=0}^1 J(\psi_k, \xi_k, \alpha_k, \Sigma_N^\varepsilon) = 0. \quad (13)$$

On $C_l^\varepsilon$, $l = 1, 2$, we use the density of $D(-\Delta) \cap C^1(\mathbb{D}) \oplus \text{span}[r^2 \sin \theta, r^2 \cos \theta]$ in $D(-\Delta)$, this allows us to write $u$ in the form (2) with $u_R(\cdot, t) \in C^1(\mathbb{D})$ for all $t \in (0, T)$, which implies that, for any $t \in (0, T)$:

$$\psi_k(\cdot, t) = O(\sqrt{\varepsilon}) \quad \text{and} \quad |\nabla \psi_k(\cdot, t)| = O\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

Using the continuity of $\alpha_k$ and $\xi_k$, one can have:

$$\lim_{\varepsilon \to 0} \sum_{k=0}^1 J(\psi_k, \xi_k, \alpha_k, C_l^\varepsilon) = 0, \quad l = 1, 2. \quad (14)$$

Then from (12)–(14), we deduce that:

$$\lim_{\varepsilon \to 0} \sum_{k=0}^1 J(\psi_k, \xi_k, \alpha_k, \Sigma^\varepsilon) = 0.$$

Finally, by Lebesgue’s theorem, we have:

$$\lim_{\varepsilon \to 0} \sum_{k=0}^1 I(\psi_k, \xi_k, \alpha_k, Q^\varepsilon) = \sum_{k=0}^1 I(\psi_k, \xi_k, \alpha_k, Q_T).$$

To achieve the proof of Theorem 2.1 we use, as in [4,6], the usual technics for Carleman estimates and the fact that $\xi_1 \leq \xi_0$ and $e^{-s\alpha_1} \leq e^{-s\alpha_0}$.

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