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An explicit counterexample for the L^p -maximal regularity problem*Un contre-exemple explicite pour le problème de la régularité maximale L^p* Stephan Fackler¹

Institute of Applied Analysis, University of Ulm, Helmholtzstraße 18, 89069 Ulm, Germany

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ABSTRACT

In this short Note we give a self-contained example of a consistent family of holomorphic semigroups $(T_p(t))_{t \geq 0}$ such that $(T_p(t))_{t \geq 0}$ does not have maximal regularity for $p > 2$. This answers negatively the open question whether maximal regularity extrapolates from L^2 to the L^p -scale.

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R É S U M É

Dans cette Note, nous démontrons l'existence d'une famille de semi-groupes holomorphes $(T_p(t))_{t \geq 0}$ telle que $(T_p(t))_{t \geq 0}$ n'a pas la régularité maximale pour $p > 2$. De cette façon, nous répondons négativement à la question ouverte qui consiste à savoir si la régularité maximale extrapole entre L^2 et L^p .

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On dit que le générateur infinitésimal $-A$ d'un semi-groupe fortement continu sur un espace de Banach X a la *régularité maximale* (pour des temps finis) si, pour tout $f \in L^2([0, T]; X)$, la solution faible $u(t) = \int_0^t T(t-s)f(s)ds$ pour le problème de Cauchy non-homogène abstrait (1) satisfait $u \in W^{1,2}([0, T]; X) \cap L^2([0, T]; D(A))$. Si $-A$ est un générateur avec la propriété de la régularité maximale, le semi-groupe engendré par $-A$ est holomorphe [5, Theorem 2.2]. Réciproquement, le générateur d'un semi-groupe holomorphe sur un espace de Hilbert possède toujours la régularité maximale [6, Lemma 3.1]. Selon un résultat de N.J. Kalton et G. Lancien, cette propriété caractérise les espaces de Hilbert dans la classe des espaces de Banach ayant une base de Schauder inconditionnelle [8]. Ils obtiennent la caractérisation en utilisant des résultats abstraits issus de la géométrie des espaces de Banach. En particulier, leur approche ne donne pas d'information sur la nature d'un contre-exemple. Dans cette Note, nous présentons une preuve plus concrète, qui montre que l'on peut choisir un contre-exemple sous la forme d'un opérateur diagonal relativement à une certaine base. De plus, le problème suivant important est resté ouvert dans l'article de Kalton et Lancien [2, 7.2.2] : dans beaucoup de cas concrets, il existe une famille de semi-groupes $(T_p(t))_{t \geq 0}$ consistants sur L^p ($1 < p < \infty$). Si le semi-groupe $(T_2(t))_{t \geq 0}$ est holomorphe, il possède la régularité maximale sur L^2 . On sait que les semi-groupes $(T_p(t))_{t \geq 0}$ sont aussi holomorphes, mais l'on ne sait pas s'ils possèdent la régularité maximale. En d'autres termes, l'holomorphie d'un semi-groupe *extrapole*, et il est naturel de se demander si la régularité maximale extrapole aussi. En fait, si le semi-groupe admet des estimations gaussiennes, il s'avère que la régularité maximale extrapole.

E-mail address: stephan.fackler@uni-ulm.de.

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L'objet de cet article est de résoudre ce problème. Ainsi, nous présentons un exemple d'une famille de semi-groupes $(T_p(t))_{t \geq 0}$, holomorphes pour tout $p \in (1, \infty)$, qui n'ont pas la régularité maximale pour $p > 2$. De plus, ces semi-groupes sont très explicites : il s'agit de multiplicateurs par rapport à une base conditionnelle de $L^p[0, 1]$.

Une version plus générale de cet article sera publiée [7] dans une autre revue. En fait, tandis que dans [7] nous considérons des espaces de Banach arbitraires qui possèdent une base de Schauder inconditionnelle, ici, ce qui simplifie considérablement les preuves, est que nous nous concentrons sur le cas des espaces L^p . De cette manière, nous espérons que les arguments sont plus accessibles aux mathématiciens moins intéressés par la géométrie des espaces de Banach que par les applications aux équations aux dérivées partielles.

1. Introduction

The generator $-A$ of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space X has *maximal regularity* (on finite time intervals) if for all $f \in L^2([0, T]; X)$ ($T \in (0, \infty)$) the mild solution $u(t) = \int_0^t T(t-s)f(s) ds$ of the inhomogeneous abstract Cauchy problem

$$\begin{cases} \dot{u}(t) + Au(t) = f(t) \\ u(0) = 0 \end{cases} \quad (1)$$

satisfies $u \in W^{1,2}([0, T]; X) \cap L^2([0, T]; D(A))$.

If $-A$ has maximal regularity, then $-A$ is the generator of a holomorphic semigroup [5, Theorem 2.2]. Conversely, all generators of holomorphic semigroups on Hilbert spaces have maximal regularity [6, Lemma 3.1]. N.J. Kalton and G. Lancien showed that this property characterizes Hilbert spaces up to isomorphism in the class of Banach spaces admitting an unconditional basis [8]. Their proof uses abstract tools from the geometric theory of Banach spaces and gives not much information on the structure of the counterexample. In the present paper we give a more direct proof which also shows that a counterexample can be given in form of a diagonal operator with respect to a basis. Another important problem remained open after the work of Kalton–Lancien [2, 7.2.2]. In many concrete cases, a family $(T_p(t))_{t \geq 0}$ of consistent C_0 -semigroups on L^p ($1 < p < \infty$) is given. One knows that the semigroup $(T_2(t))_{t \geq 0}$ is holomorphic, hence it has maximal regularity on L^2 . One knows that then also $(T_p(t))_{t \geq 0}$ is holomorphic, but the question is whether $(T_p(t))_{t \geq 0}$ also has maximal regularity. In other words, it is known that holomorphy is a property which *extrapolates* (from L^2 to L^p) and the question is whether the same is true for maximal regularity. In fact, one might come to this conjecture since the answer is positive if the semigroup has Gaussian estimates.

The aim of this article is to show that nonetheless the answer is negative in general. In fact, we will give a counterexample in form of semigroups given as multiplication operators with respect to a Schauder basis of $L^p[0, 1]$ which act consistently on all L^p ($1 < p < \infty$) such that $(T_2(t))_{t \geq 0}$ is holomorphic but $(T_p(t))_{t \geq 0}$ has not maximal regularity for any $p > 2$.

A more extended version of this article will appear elsewhere [7]. In fact, whereas in [7] we consider an arbitrary Banach space with an unconditional basis, here we merely consider L^p -spaces, which simplifies the proofs. In this way we hope that the arguments are accessible to mathematicians who are less interested in geometry of Banach spaces and more in applications to PDE.

2. The counterexample

On $L^p[0, 1]$ ($p \in (1, \infty)$) one has the following equivalent characterization for the maximal regularity (on finite time intervals) of the holomorphic semigroup $(T(t))_{t \geq 0}$ generated by $-A$ obtained by L. Weis [12]: there exists a $\delta \in (0, \frac{\pi}{2}]$ such that for arbitrary z_1, \dots, z_n in $\Sigma_{\delta,1} := \{z \in \mathbb{C} \setminus \{0\} : |z| \leq 1, |\arg z| < \delta\}$ and all $f_1, \dots, f_n \in L^p[0, 1]$ one has:

$$\int_0^1 \left\| \sum_{k=1}^n r_k(\omega) T(z_k) f_k \right\|_{L^p[0,1]} d\omega \leq C \int_0^1 \left\| \sum_{k=1}^n r_k(\omega) f_k \right\|_{L^p[0,1]} d\omega$$

for some constant $C \geq 0$, where $r_k(\omega) = \text{sign} \sin(2^k \pi \omega)$ are the Rademacher functions. In other words, the semigroup has maximal regularity if and only if there exists a $\delta > 0$ such that the set $\{T(z) : z \in \Sigma_{\delta,1}\}$ is \mathcal{R} -bounded. Proofs of the above facts and details about maximal regularity can be found in the references [9] and [3]. We use this to give the following counterexample.

Theorem 2.1. *There exist consistent holomorphic C_0 -semigroups $(T_p(z))_{z \in \Sigma_{\pi/2}}$ on $L^p[0, 1]$ for $p \in (1, \infty)$ such that $(T_p(z))_{z \in \Sigma_{\pi/2}}$ does not have maximal regularity for $p \in (2, \infty)$.*

2.1. Proof

Consider the normalized Haar system $(h_n)_{n \in \mathbb{N}}$ which is an unconditional basis of $L^p[0, 1]$ for $p \in (1, \infty)$. Choose a subsequence $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that the functions h_{n_k} have pairwise disjoint supports. Then $(h_{n_k})_{k \in \mathbb{N}}$ is an unconditional basic

sequence equivalent to the standard basis in ℓ^p . Choose a permutation of the even numbers such that $\pi(4k) = n_k$. We now define:

$$e_n := \begin{cases} h_{\pi(n)} & n \text{ odd} \\ h_{\pi(n)} + h_{\pi(n-1)} & n \text{ even.} \end{cases}$$

As a block perturbation of the basis $(h_{\pi(n)})_{n \in \mathbb{N}}$, $(e_n)_{n \in \mathbb{N}}$ is a basis for $L^p[0, 1]$ as well [10, Proposition 4.4]. Further, let A be the closed linear operator on $L^p[0, 1]$ given by:

$$D(A) = \left\{ x = \sum_{n=1}^{\infty} a_n e_n : \sum_{n=1}^{\infty} 2^n a_n e_n \text{ exists} \right\}$$

$$A \left(\sum_{n=1}^{\infty} a_n e_n \right) = \sum_{n=1}^{\infty} 2^n a_n e_n.$$

Since $(2^n)_{n \in \mathbb{N}}$ is increasing, $-A$ generates consistent holomorphic C_0 -semigroups $(T_p(z))_{z \in \Sigma_{\pi/2}}$ on $L^p[0, 1]$ for all $p \in (1, \infty)$ [11, Theorem 3.2], which are given as multipliers associated with the sequences $(e^{-2^n z})_{n \in \mathbb{N}}$.

For $p \in (2, \infty)$ the basic sequences $(h_{\pi(4k)})_{k \in \mathbb{N}}$ and $(h_{4k+1})_{k \in \mathbb{N}}$ are not equivalent, as the latter possesses a normalized block basic sequence for which the coordinates of every expansion necessarily lie in ℓ^2 (choose each block as the sum of the Haar functions whose supports have identical Lebesgue measure and use the first part of the proof of [10, Proposition 21.1]), whereas every seminormalized block basic sequence of the former is equivalent to the standard basis in ℓ^p [1, Remark 2.1.2]. In particular, there exists a sequence $(a_k)_{k \in \mathbb{N}}$ which converges with respect to $(h_{\pi(2k)})_{k \in \mathbb{N}}$, but not with respect to $(h_{2k+1})_{k \in \mathbb{N}}$. Now, for $p \in (2, \infty)$ assume that $(T_p(t))_{t \geq 0}$ is \mathcal{R} -bounded on $[0, 1]$. Then for some fixed sequence $(q_n)_{n \in \mathbb{N}} \subset (0, 1)$:

$$\mathcal{T} : \sum_{k=1}^N r_k x_k \mapsto \sum_{k=1}^N r_k T(q_k) x_k$$

extends from the finite Rademacher sums to a bounded linear operator on the closure in $L^1([0, 1]; L^p[0, 1])$ denoted by $\text{Rad}(L^p[0, 1])$. Now consider $x = \sum_{k=1}^{\infty} a_k r_k h_{\pi(2k)}$ which converges in $\text{Rad}(L^p[0, 1])$ by the unconditionality of the basis. Then the boundedness of \mathcal{T} shows that:

$$\begin{aligned} \mathcal{T}x &= \lim_{N \rightarrow \infty} \mathcal{T} \left(\sum_{k=1}^N a_k r_k (e_{2k} - e_{2k-1}) \right) = \lim_{N \rightarrow \infty} \sum_{k=1}^N a_k r_k (e^{-2^{2k} q_k} e_{2k} - e^{-2^{2k-1} q_k} e_{2k-1}) \\ &= \lim_{N \rightarrow \infty} \sum_{k=1}^N a_k r_k e^{-2^{2k} q_k} h_{\pi(2k)} + a_k r_k (e^{-2^{2k} q_k} - e^{-2^{2k-1} q_k}) h_{2k-1} \end{aligned}$$

converges in $\text{Rad}(L^p[0, 1])$ which implies convergence almost everywhere [4, Theorem 12.3]. In particular, for the choice $q_k = \frac{\log 2}{2^{2k-1}}$ (which maximizes the factor in the second summand) and applying the coordinate functionals $(h_{2k-1}^*)_{k \in \mathbb{N}}$, we see that for almost all $\omega \in [0, 1]$:

$$\frac{1}{4} \sum_{k=1}^{\infty} r_k(\omega) a_k h_{2k-1} \quad \text{and therefore} \quad \sum_{k=1}^{\infty} a_k h_{2k-1}$$

converge by the unconditionality of the Haar basis. This, however, contradicts our choice of $(a_k)_{k \in \mathbb{N}}$.

Remark 1. By using the theory of symmetric bases one can give a new proof of the Kalton–Lancien result along the lines of the above argument. Furthermore, by considering consistently chosen direct sums of the above semigroups with their adjoint semigroups one obtains a family of consistent C_0 -semigroups $(T_p(t))_{t \geq 0}$ ($p \in (1, \infty)$) such that $(T_p(t))_{t \geq 0}$ has maximal regularity if and only if $p = 2$. We refer to the preprint [7] for details.

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